University of California
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On Stochastic Decentralized Systems in Communications and Control

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

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2001

To my father and my mother,

to my dearest Maryam,

and to my dear brothers and sisters,

for their continuous encouragement and support ...

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Abstract of the Dissertation

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It is known that higher efficiency, reliability and fault tolerance can be achieved in decentralized systems, where multiple, largely autonomous, components are integrated. While continuous advancements in computer, communication, and control technologies have expanded the possibilities for creating such systems, there are still some fundamental issues in designing distributed algorithms that achieve a prescribed level of performance. The objective of our research was to address some of these issues with more emphasis on the control and communication aspects.

In the first part, we address the notion of control-oriented value of information. Considering Linear Quadratic Gaussian (LQG) systems, we show how the measurements can be evaluated based on their effect on the performance and how information measures may be incorporated in a given performance index. Furthermore, we investigate how communicating various pieces of information among the local stations in a decentralized system may affect the overall performance and specifically the global stability of the decentralized system.

In the second part, we consider the Witsenhausen counter-example, which is a simple two-stage stochastic decentralized system with a non-classical information pattern. We present various reformulations of the problem and elaborate on the difficulties involved in designing optimal strategies. Assuming that the two stations communicate through a low noise channel, we show, through an asymptotic analysis, that the linear strategies still satisfy the necessary condition for optimality.

In the final part, we focus on power control problem, which can be regarded as a stochastic decentralized regulation problem in cellular wireless systems. We unify the two main approaches for information-feedback distributed power control design and obtain an insightful sufficient condition for network feasibility. Moreover, we use a robust control framework for global stability analysis of a power-controlled network. We then design a Kalman predictive distributed power control algorithm. We show, through extensive system-level simulations, that under the dynamics of user arrivals and departures and user mobility, significant improvement in performance can be achieved when our predictive power control algorithm is integrated with a distributed minimum interference dynamic channel assignment scheme.

Part I

Preliminaries

CHAPTER 1

Introduction

Driven by the continuous advancements and innovations in computer, communication, and control technologies, possibilities for design and implementation of highly efficient and reliable engineering systems have been steadily increasing. To achieve a high level of efficiency, reliability, and fault tolerance, complex systems should be designed based on decentralized structures, where multiple, largely autonomous, components are integrated. At the same time, many new applications are emerging, where inherently decentralized systems are present. Such systems are usually composed of a large number of complex, possibly spatially distributed, stations or agents that are interacting with each other.

The basic centrality assumption, which is prevalent among classical engineering approaches, fails to hold in such large scale systems. In fact, one of the main characteristics of these systems is that distributed decisions must be made based on decentralized information. In centralized systems, all control actions (decisions) are taken by one controller (agent) in one station where all the information is gathered, whereas in most large scale systems there are several control stations that have access to different pieces of information. These stations may communicate with each other possibly by signaling through noisy channels. They usually have to coordinate their strategies in order to achieve a common objective. Decentralization of information and capability of communication among the stations make these systems drastically different from centralized systems and pose many great challenges to the system engineers.

Decentralized systems have been addressed in a very wide range of applications. Some examples are:

- Communication Networks
- Coordinated Autonomous Vehicles
- Flexible Manufacturing Systems
- Economic Systems
- Distributed Power Systems
- Distributed Database Systems

Due to the diversity of applications, a lot of research effort has been devoted to studying decentralized systems, in different areas and at various levels of abstraction, during the past decades. This includes research in Team Theory and Mathematical Economics, Control Theory, and Information Theory. Some attempts have also been made towards developing a unified theory to analyze such systems. However, despite all these efforts, there are still many fundamental difficulties in dealing with these systems. The main objective of our research was to address some of these difficulties.

In Section 1.1, we explain the direction of our research and provide the outline of this thesis. This outline serves as an extended summary and also as a guideline on how the different chapters are related. It thus helps the readers find their topic of interest more easily. Then, in Section 1.2, we provide a summary of our main contributions.

1.1 Thesis Outline

We introduce some of the basic concepts in decentralized systems and team theory in Chapter 2. We start by reviewing the main components of a team problem. We then show how a decentralized control problem may generally be formulated and how it relates to a team problem. We review the notion of information pattern and discuss different types of information patterns and their implications for designing decentralized control algorithms. Finally, we mention some applications in control and communication systems. Among these applications, power control for cellular radio systems constitutes a major focus of our research.

The first part of the thesis discusses the notion of the *value of information* and how the transmission of information among various stations can affect the performance in a decentralized control system.

The concept of the value of information was, in fact, the starting point for our research. While the value of information for transmission is a well-established concept in information theory, the value of information for control is still an outstanding issue. Note that the best achievable performance in any decentralized system is greatly affected by the prevailing information pattern in that system, i.e., *who knows what and when.* Therefore, there is a need to find a measure for evaluating a piece of information, based on how it affects the performance objective. Unfortunately, such a measure would highly depend on the specific structure of the performance index and obtaining a unified measure seems out of reach. Nevertheless, we decided to review some possible approaches.

We introduce the notion of the value of information in Chapter 3. We then discuss the *entropy* approach to estimation and control. While this approach was mainly proposed to explain the dual control effect in adaptive control algorithms, it can be seen as a reasonable platform for evaluation of different pieces of information for control purposes. We continue by investigating a classical Linear Quadratic Gaussian (LQG) control problem. We explain how the effect of different measurements on the performance might be evaluated simply by looking at their corresponding noise intensities. Finally, we propose a scheme where a measure of information can directly be incorporated in a quadratic performance index.

In Chapter 4 we consider a simple decentralized LQG problem where the stations are allowed to communicate different pieces of information, such as their measurements or their control values. We propose a *sub-optimal* approach where the control algorithm is obtained by solving separate centralized LQG problems. We focus on the closed-loop stability of the global system, under various communication scenarios, and show how the closed-loop system may become unstable even when the stations communicate all their measurements.

In the second part of the thesis, we consider a two-stage stochastic decentralized optimal control problem. This is based on a classical example, proposed by Witsenhausen in 1968, where, despite the linear dynamics, a quadratic cost, and additive and Gaussian uncertainties, the non-classical nature of the information pattern transforms the problem into a non-convex functional optimization problem.

In Chapter 5 we state the problem and then consider its various reformulations. Namely, we will investigate the scenarios in which the two stations are allowed to communicate through noiseless or noisy channels. We will see that noiseless transmission of information among the stations changes the problem into a trivial one, whereas any noise in transmission results in a non-classical information pattern again. Finally, we review a reformulation of the problem as a communications problem, where concepts from information theory are used to obtain the optimal strategies.

In Chapter 6 we look at a very interesting special case. Namely, we assume that the uncertainty in communication among the two stations is small. We follow an asymptotic approach to obtain an expansion for the performance index. We then use a variational approach to obtain a necessary condition for the asymptotically optimal strategies. Finally, we show how the linear strategies, with slightly different coefficients than the noiseless transmission case, satisfy the necessary condition for asymptotic optimality.

The next part of the thesis focuses on a very specific application in wireless communications. Namely, we investigate the power control problem in cellular radio systems. In a cellular system, a single channel can be shared by multiple users. The objective is to design a decentralized or distributed algorithm to control the transmit power levels of the users (uplink) and the base stations (downlink) in order to achieve the required level of quality of service for every user, with the minimum possible power, while eliminating unnecessary interference on other co-channel users.

In Chapter 7 we explain the problem along with the main approaches that have been proposed in the literature. Specifically, we discuss Signal to Interference plus Noise Ratio (SIR) balancing and SIR threshold approaches and try to unify them. Moreover, we explain how the power control problem can be formulated as a decentralized regulation problem and how the global stability of the network may be analyzed.

We propose a novel decentralized predictive power control algorithm in Chapter 8. It is shown how simple models for the slow variations in the channel gains and interference levels may be incorporated in simple Kalman predictors. We also analyze the global stability of the network, on a single channel, under our predictive power control algorithm.

In order to compare our predictive power control algorithm with the one that uses no prediction, under more realistic scenarios, we decided to set up a general system-level simulation platform. This platform can be used to compare various integrated Dynamic Channel and Power Allocation (DCPA) schemes, under the dynamics of user arrivals and departures and user mobility.

In Chapter 9 we review Dynamic Channel Assignment (DCA) schemes and then provide the details of our simulation platform. We then discuss our simulation results and show the improvement in performance when our predictive power control algorithm is integrated with a minimum interference DCA scheme.

The last part of the thesis includes the concluding remarks and some directions for future research.

1.2 Summary of Contributions

As we mentioned, the main objective of this research was to address some of the major difficulties in dealing with decentralized stochastic systems. We started by looking at the concept of the value of information and then moved towards a decentralized LQG problem. We then focused on the reformulations of a two-stage decentralized stochastic system with a non-classical information pattern. Finally, we investigated the power control problem as a specific application in wireless communications. We addressed some similar issues that come up in designing decentralized power control algorithms for cellular networks.

The main contributions of our research are listed below by chapter:

- Chapter 3:

- Proposed the concept of critical versus non-critical measurements in an LQG control problem, based on detectability of the system.
- Proposed a scheme to evaluate non-critical measurements in an LQG control problem, based on their corresponding noise covariances.
- Proposed a scheme to directly incorporate covariance-dependent information cost into the original quadratic performance index.

- Chapter 4:
  - Proposed a sub-optimal design for a two-station decentralized LQG problem, where the control strategies are obtained by solving separate centralized LQG problems.
  - Analyzed the global closed-loop stability under various communication scenarios between the stations. Namely, when the stations communicate their estimates or control values, their measurements, and their estimation residuals.
  - Showed that even when the stations communicate all their measurements, the sup-optimal controllers fail to stabilize the global closed-loop system, if the compensators have unstable dynamics.
  - Showed how communicating control values in addition to the measurements can help in stabilizing the closed-loop system.
  - Showed that if the communication uncertainties are small, transmission of estimation residuals can replace transmission of measurements and control values.

- Chapter 5:

- Proposed a reformulation of the Witsenhausen counter-example, where the first station is allowed to send its information to the second station through noisy channels.
- Obtained an alternative form of the performance index. Namely, the cost is expressed only in terms of a single strategy, using a Fisher information term.
- Considered the two limit cases where the transmission noise intensity goes to zero or grows to infinity. It is shown that the reformulated example covers a wide range, from a classical LQG problem to Witsenhausen's counter-example.

• Chapter 6:
  - Considered a special case of the reformulated example in Chapter 5, where the transmission noise intensity is small.
  - Used an asymptotic approach in order to obtain an expansion for the cost, in terms of the small transmission noise intensity.
  - Using a variational (Hamiltonian) approach, obtained a necessary condition for asymptotically optimal strategies.
  - Showed that the linear strategies, with slightly different coefficients than the noiseless transmission case, do indeed satisfy the necessary condition for asymptotic optimality.

• Chapter 7:
  - Unified SIR balancing and SIR threshold approaches for power control design.

  - Obtained an insightful sufficient condition, based on individual channel gains and desired SIR thresholds, for network feasibility.
  - Formulated power control as a decentralized regulation problem and analyzed the global stability of the network.
  - Using a robust control framework, obtained a sufficient condition for global stability of the network in $\ell_\infty - induced$ norm sense.

- Chapter 8:
  - Proposed simple models for the slow variations in the channel gains and interference plus noise levels for every user.
  - Designed a predictive distributed power control algorithm, based on simple Kalman predictors.
  - Analyzed the global stability of the network, on a single channel, under the predictive power control algorithm.

- Chapter 9:
  - Developed a general system-level simulation platform for comparing various Dynamic Channel and Power Allocation schemes for a Time-Frequency Division Multiple Access system, under the dynamics of user arrivals and departures and user mobility.
  - Simulated the integrated predictive power control and minimum interference Dynamic Channel Assignment, and compared with a simple integrator DCPA scheme with no prediction.
  - Analyzed the simulation results and showed the improvement in performance when the predictive power control algorithm is employed.

CHAPTER 2

Team Theory, Decentralized Systems, and Information Patterns

2.1 Introduction

Research on multi-agent decision making processes was initiated in the mathematical economics and was formalized as the team theory, mainly by the work of Marschak and Radner who developed the concepts for the theory of a firm in the 1950s and early 1960s [55, 61, 62]. The connections between team theory and decentralized control theory were mostly explained in the work of Witsenhausen, Ho and others in late 1960s and early 1970s [36, 84, 85].

Since then, new applications in many different areas have emerged, where the objective is to control a large-scale system, which is composed of multiple interconnected and highly autonomous local stations. Our goal in this chapter is to introduce some of the basic concepts in team theory and decentralized systems.

In Section 2.2, we define a *team* and explain the basic components of any team decision problem. We then introduce the notion of *person-by-person* optimality and describe an important result for Linear Quadratic Gaussian teams. In Section 2.3 we formulate a general decentralized control problem and explain the connections between decentralized control theory and team theory.

Another important concept for any team or any decentralized system is *infor-*

mation pattern, which determines the local information available at any station and at any given instant of time. The information pattern can have a tremendous effect on the optimal control strategies and thus on the overall performance in a decentralized system. In Section 2.4 we explain two important classifications for information patterns and discuss their implications on the control design.

To get a better feeling for the theoretical concepts, we briefly review a few specific applications in Sections 2.5 and 2.6. Namely, we look at the formation flight and truck platoons as two application examples in control. Then we describe rate-based congestion control in ATM networks and power control in cellular wireless systems as two application examples in communications. Power control, in fact, turns out to be a major focus of our research.

2.2 Team Theory

A *team* is defined as a group of agents or decision makers who act in a coordinated manner, in an uncertain environment, in order to achieve a common goal. These agents usually have access to different, but correlated, information about the underlying uncertainties. This imposes a decentralized nature on the process of decision making. Team theory is the study of such situations with prevalent applications in large scale systems.

There are five basic ingredients for any team decision problem [37, 38]:

1. A vector of random variables $\xi = \begin{bmatrix} \xi_1, & \ldots, & \xi_m \end{bmatrix} \in \Xi$, defined on a probability space with a given density $p(\xi)$. This vector includes all the underlying uncertainties in the system, such as unknown initial conditions, measurement noise, disturbances, etc. It is often called *"state of the world"* or *"state of the nature"*.

2. A set of *decision variables* $u = \begin{bmatrix} u_1, & \ldots, & u_n \end{bmatrix} \in U$, which represents n agents or decision makers. Note that there is no loss of generality in assuming scalar decision variables and associating each with one agent. Vector decision variables, or agents who make different decisions at different times, can both be decomposed into more agents with scalar decision variables.

3. A set of *measurements* or *observations* $z = \begin{bmatrix} z_1, & \ldots, & z_n \end{bmatrix} \in Z$, which represents the information available to different agents. Each observation is generally expressed by a given vector functions $z_i = \eta_i(\xi), i = 1, \ldots, n$. In a more general setting, where the order of agents' actions becomes important, each z_i could be a function of both ξ and u, subject to the requirements of causality and deadlock avoidance. The η_i's represent the *information pattern* of the system, which will be discussed in more detail, later in this chapter.

4. A set of *strategies* (*control laws, decision rules*) $\gamma = \begin{bmatrix} \gamma_1, & \ldots, \gamma_n \end{bmatrix} \in \Gamma$, where each γ_i is a mapping from the z_i-space to the u_i-space, to reflect the fact that each agent makes its own decision based on its own local information.

5. A cost (or payoff) function L, which is a measurable function of u and ξ.

When $z_i = \eta_i(\xi)$, and the strategies are given, $L(u, \xi) = L(\gamma(\eta(\xi)), \xi)$ is a well-defined random variable (provided that γ and η are appropriately measurable functions). Therefore, its expectation is also well-defined and the team problem is stated as follows:

$$\min_{\gamma \in \Gamma} J(\gamma) = \min_{\gamma \in \Gamma} E_\xi \left[L(\gamma(\eta(\xi)), \xi) \right]. \tag{2.1}$$

Note that J is a functional and this is, in fact, a function optimization problem over the Γ space, which may have very little structure. Also note that a team problem is a cooperative problem in the sense that each agent is allowed to know the strategies of all other agents beforehand. In other words, while the observations are decentralized, the strategies are known to all agents in the team.

A necessary condition for optimality of a set of strategies γ is that each strategy be optimal when all other strategies are fixed. This is called *person-by-person optimality*. Let $\bar{\gamma}_i$ denote the strategies of all other agents except agent i, and assume it is fixed and known to the i-th agent. Then we have:

$$\begin{aligned}\min_{\gamma_i\in\Gamma_i} J\left(\gamma_i,\bar{\gamma}_i\right) &= \min_{\gamma_i\in\Gamma_i} E_{z_i}\left[E_\xi\left[L\left(\gamma_i,\bar{\gamma}_i,\xi\right)|z_i\right]\right] \\ &= E_{z_i}\left[\min_{u_i\in U_i} E_\xi\left[L\left(u_i,\bar{\gamma}_i,\xi\right)|z_i\right]\right],\end{aligned} \tag{2.2}$$

where we have used the following *fundamental lemma* [56]:

$$\min_{u(.)} E_x\left[f\left(x,u(x)\right)\right] = E_x\left[\min_u f(x,u)\right]. \tag{2.3}$$

Note that $\min_{u_i\in U_i} E_\xi\left[L\left(u_i,\bar{\gamma}_i,\xi\right)|z_i\right] = \min_{u_i\in U_i} J_i\left(u_i,z_i,\bar{\gamma}_i\right)$ poses a parameter optimization problem for each agent. Person-by-person optimality is always a necessary condition for team optimality. But it is sufficient only if the cost function L is *convex* in all the strategies. This is because convexity implies:

$$L\left(u,\xi\right) \geq L\left(u^*,\xi\right) + \nabla_u L^T\left(u^*,\xi\right)\cdot\left(u-u^*\right). \tag{2.4}$$

Let u^* denote the person-by-person optimal set of strategies. Then:

$$\begin{aligned}
J(\gamma) &= E\left[L\left(u=\gamma(z), \xi\right)\right] \\
&\geq J(\gamma^*) + E\left[\nabla_u L^T(u^*, \xi) \cdot (\gamma - \gamma^*)\right] \\
&= J(\gamma^*) + \sum_{i=1}^{n} E\left[(\gamma_i - \gamma_i^*) \left.\frac{\partial L}{\partial u_i}\right|_{u^*}\right] \\
&= J(\gamma^*) + \sum_{i=1}^{n} E_{z_i}\left[(\gamma_i - \gamma_i^*) E_\xi\left[\left.\frac{\partial L}{\partial u_i}\right|_{u^*} \middle| z_i\right]\right] \\
&= J(\gamma^*) + \sum_{i=1}^{n} E_{z_i}\left[(\gamma_i - \gamma_i^*) \left.\frac{\partial}{\partial u_i} E_\xi\left[L \,|z_i\right]\right|_{u^*}\right] \\
&= J(\gamma^*),
\end{aligned} \tag{2.5}$$

where we have used the fact that u^* is the set of person-by-person optimal strategies and hence from (2.2), we have:

$$\left.\frac{\partial}{\partial u_i} E_\xi\left[L \,|z_i\right]\right|_{u^*} = 0. \tag{2.6}$$

We can now state an important classic result in team theory:

Theorem 1. *[38] In Linear Quadratic Gaussian (LQG) teams, where:*

$$L = \frac{1}{2} u^T Q u + u^T S \xi, \quad Q > 0, \quad \xi \sim \mathcal{N}(0, \Sigma) \tag{2.7}$$

$$z = H\xi, \tag{2.8}$$

the person-by-person optimal solution is linear in the information z and is the unique team optimal solution.

Note that the information available for every station only depends on the underlying uncertainties in the system. As we shall see in Section 2.4, this is called a *static* information pattern.

2.3 Decentralized Systems

Any centralized control problem involves a dynamic system that is evolving with time according to a set of ordinary or partial differential or difference equations. A set of control strategies must be designed so that each control input is generated using *all* the information available in the system. The notions of time and of a dynamic system are still the key characteristics of a decentralized control problem, even though each control input in this case, is obtained by using only the local information. This is one of the basic differences between a decentralized control problem and a team decision problem. In other words, a team problem can be seen as a time snapshot of a decentralized control system.

We now formulate a general decentralized stochastic control problem in discrete time [85]. The dynamic system is governed by the following difference equation:

$$x_{t+1} = f_t\left(x_t, u_t^1, \ldots, u_t^n, w_t\right), \quad t = 0, \ldots, T, \tag{2.9}$$

where x_t is the global state vector of the system at time t, u_t^i is the control input of the i-th controller at time t, w_t represents the process noise sequence, n is the number of control stations and T is the final time instant. Each station has its own observation, which is given by the following equation:

$$z_t^i = g_t^i\left(x_t, v_t^i\right), \quad i = 1, \ldots, n \ \text{ and } \ t = 0, \ldots, T, \tag{2.10}$$

where v_t^i is the measurement noise sequence at the i-th station. The objective is to design the strategies for the controllers to minimize the following cost function:

$$J = E\left[L\left(u_0^1, \ldots, u_0^n, \ldots, u_T^1, \ldots, u_T^n, w_0, \ldots, w_T, x_0\right)\right]. \tag{2.11}$$

The controllers can use whatever information is available to them, which includes their own observations and the information that is being communicated among the stations either through the noisy channels or through the system dynamics. Therefore, the next stage is to determine the information available to each controller, i.e., the *information pattern* of the system.

The total information available inside the system at time instant t, after all the stations have taken their measurements, can be characterized by the following two sets:

$$\mathbf{Z}_t = \left\{z_1^1, \ldots, z_1^n, \ldots, z_t^1, \ldots, z_t^n\right\} \tag{2.12}$$

$$\mathbf{U}_{t-1} = \left\{u_0^1, \ldots, u_0^n, \ldots, u_{t-1}^1, \ldots, u_{t-1}^n\right\}. \tag{2.13}$$

The information pattern of the system can be specified by assigning two information sets $\mathbf{Z}_t^i$ and $\mathbf{U}_{t-1}^i$ to every station at every time instant t, where:

$$\mathbf{Z}_t^i \subseteq \mathbf{Z}_t \quad i = 1, \ldots, n \text{ and } t = 0, \ldots, T \tag{2.14}$$

$$\mathbf{U}_{t-1}^i \subseteq \mathbf{U}_{t-1}. \tag{2.15}$$

It is clear that when we have more than one station and we are considering only one time step ($T = 0$), we indeed have a team problem.

2.4 Information Patterns

As we said earlier, the way that information is distributed in a decentralized system greatly affects the performance of the controlled system. Changes in the information pattern will produce changes in the optimal achievable cost. Even though there are always some constraints on how the information can be distributed in a physical system (where to put the sensors and the actuators, what to transmit, etc.), in general, there are many possible information patterns for a given system. Therefore, designing the *optimal information patterns* should be addressed along with designing the *optimal strategies*, in order to get the best possible performance. Note that different information patterns may yield different optimal strategies; thus, for a given set of constraints, carefully designing the pattern could lead us to noticeably better performance.

In order to analyze the effect of information availability in a decentralized system, we need to study two related problems. First, we need to investigate the structure of different information patterns and how they could affect the optimal strategies. Second, we need to see how valuable a piece of information could be. We want to know what would be a reasonable cost for installing a sensor or a transmitter in order to obtain or communicate a piece of information respectively. In other words, we need to find a measure of information based on how it affects our performance objective. We discuss different information patterns in this section. Approaches towards finding a measure of information will be discussed later in Chapter 3.

2.4.1 Static versus Dynamic Patterns

As we mentioned, the information available to the agents in a team may only depend on the underlying uncertainties in the system, i.e., $z = \eta(\xi)$. In this case, the information available to one agent is not affected by the decisions made by other agents. We call this information pattern a *static* pattern. From Theorem 1, we know that the optimal set of strategies for an LQG team with a static pattern is unique and linear in the information.

Sometimes an order may be assigned for the agents to make their decisions. In these systems, the information available to one agent can depend on the previous agents' decisions, in which case we have $z = \eta(u, \xi)$, subject to some obvious causality conditions.

In other systems, the order for the agents could even depend on the uncertainties in the system, either directly or through agents' decisions. In these systems, which are called *non-sequential systems*, it may be impossible to order the agents *a priori* and independently of the set of strategies. The information available to one agent would also depend on other agents' decisions, while some conditions should be imposed to prevent *deadlocks*, where two or more agents become mutually dependent [73].

These information patterns are called *dynamic* patterns. There is no known solution for teams with general dynamic information patterns. In general, the information available to one agent may be insufficient to determine the previous decisions of other agents, which have affected that information [85]. This, in a sense, would make that piece of information less valuable.

In some cases, however, even though the pattern is dynamic, each agent knows or can reconstruct the decisions of other agents who have affected its own information. This is a special case of a dynamic pattern, which is called a *partially nested pattern* [39]. For example, assume that the information available to u_3 is affected by u_1 and not by u_2. We will have a partially nested pattern if u_3 knows or can reconstruct what u_1 did, regardless of whether or not it knows what u_2 did. It can be shown [39] that a dynamic LQG team with a partially nested pattern can be converted into a static LQG team and hence has a unique optimal strategy, which is linear in the information.

Using the above characterization for information patterns, we see that in most cases, decentralized control problems have dynamic patterns. This is because of the presence of a dynamically evolving state vector, which is affecting the information available to the controllers as time passes, while it is being affected itself by the control actions.

2.4.2 Classical versus Non-classical Patterns

There is another classification for information patterns that is more applicable to decentralized control systems. From (2.12) to (2.15), we recall that the information pattern in a decentralized control system would be specified by assigning the information sets $\mathbf{Z}_t^i$ and $\mathbf{U}_{t-1}^i$ to every i-th station at every time instant t.

We say the i-th control station has *perfect recall* if it does not lose any information as time passes ($T = 0$ is the trivial case), that is:

$$\mathbf{Z}_t^i \subset \mathbf{Z}_{t+1}^i \tag{2.16}$$

$$\mathbf{U}_{t-1}^i \subset \mathbf{U}_t^i \,. \tag{2.17}$$

An information pattern is called *classical* if all control stations have access to the same information and they all have perfect recall. Otherwise, it is called a *non-classical* pattern.

It is clear that in any system with a classical information pattern, all control stations can be combined into one control station and we can deal with the system as if it were centralized [85]. Hence all the classic results from the centralized control theory hold for such systems. Mainly, if the system is linear and the process noise and the measurement noise sequences are assumed to be Gaussian, then the *separation principle* holds, i.e., the optimal control can be represented as a function of the conditional density of the state:

$$u_t^* = \gamma_t^* \left(\mathbf{Z}_t, \mathbf{U}_{t-1}\right) = \phi_t \left(p\left(x_t \left|\mathbf{Z}_t, \mathbf{U}_{t-1}\right.\right)\right) \tag{2.18}$$

where $p\left(x_t \left|\mathbf{Z}_t, \mathbf{U}_{t-1}\right.\right)$ is the conditional density of x_t given the information. In this case, we actually know that the optimal control only depends on the conditional mean of x_t given the information. Moreover, when the cost is also quadratic, the *certainty equivalence principle* will hold as well, i.e., the optimal control at time t will be obtained by replacing x_t with its estimate in the optimal control for the corresponding deterministic system, where all the uncertainties are replaced by their mean values. Also, as we know, in this LQG case, the optimal control is indeed linear in the information.

All these properties fail to hold when the decentralized system has a general non-classical information pattern. We now mention some special forms of non-classical patterns, which have been studied in the literature.

One important class of such patterns is the class of *k-step-delayed sharing patterns.* In these patterns, the information sets of every i-th station have the following form:

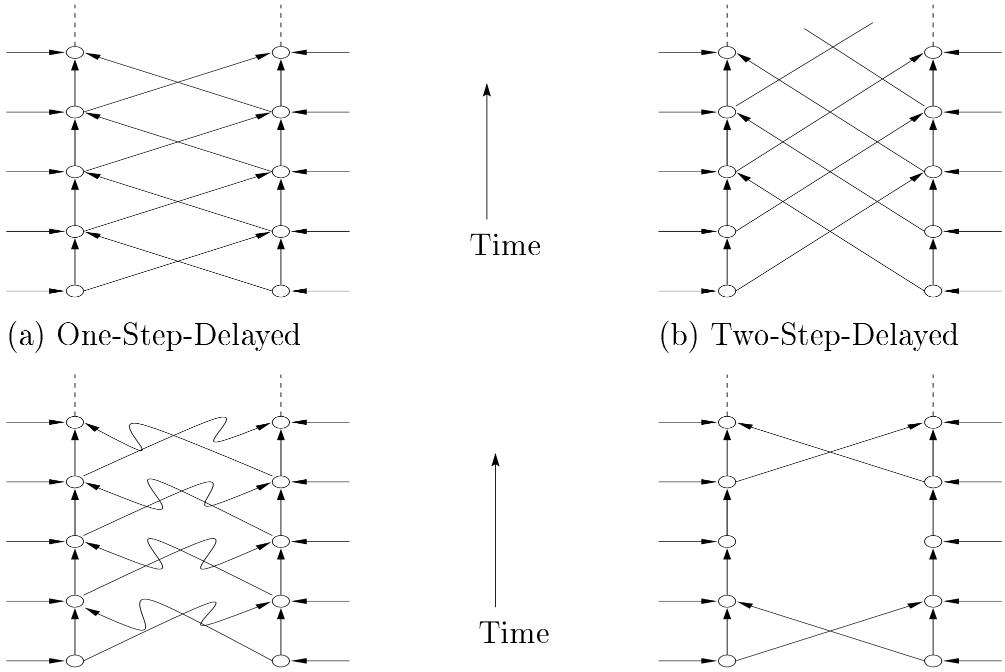
$$\mathbf{Z}_t^i = \mathbf{Z}_{t-k} \cup \left\{ z_{t-k+1}^i, \ldots, z_t^i \right\} \tag{2.19}$$

$$\mathbf{U}_{t-1}^i = \mathbf{U}_{t-k} \cup \left\{ u_{t-k+1}^i, \ldots, u_{t-1}^i \right\}, \tag{2.20}$$

where $\mathbf{Z}_{t-k}$ and $\mathbf{U}_{t-k}$ represent the information shared by all the stations. In other words, every station has access to other stations' information only after a delay of k steps. This delay could represent the communication delay among the control stations.

It is shown in [77] that the separation principle holds only if $k = 1$, that is, for the *one-step-delayed sharing pattern.* For the LQG case with this pattern, the optimal control is shown to be linear [63]. These results are also generalized in [23] for the case where the cost is an exponential of a quadratic function, i.e., the Linear Exponential Gaussian case.

In order to give a simple qualitative explanation for these results, we look at a system with only two stations. Figure (2.1) shows four different patterns. As we can see in figure (2.1.a), when the delay is only one step, each station has access to the previous control action of the other station that has affected its current information. But when the delay becomes more than one step, as shown in figure



(a) One-Step-Delayed

(b) Two-Step-Delayed

(c) One-Step-Delayed with Noise

(d) One-Step-Delayed Periodic

Figure 2.1: Information Patterns

(2.1.b), neither station knows what the other station did during the intermediate steps, even though that has affected its information through the state vector. In other words, the one-step-delayed sharing pattern has a similar characteristics as the partially nested patterns for teams, and this could explain why for the LQG case, the separation principle holds and the optimal strategies are still linear in the information.

Another situation of more practical interest is when the information is getting corrupted by noise while being communicated among the stations. In this case, even with only one step delay, which is shown in figure (2.1.c), there is no known solution for the optimal strategies. This is again because of the fact that one station does not have access to other stations' previous control actions that have affected its present information.

More recently, a *periodic sharing pattern* was introduced in [58]. In this pattern, the information is shared among the stations every m time steps. The information transfer could also be delayed for $k (k < m)$ time steps. For this pattern, the optimal strategies are not known, even though a form of separation principle has been derived in [58] for one-step-delayed case. A one-step-delayed three-step-periodic pattern for two stations is shown in figure (2.1.d).

2.5 Application Examples in Control

As we mentioned, there are many applications where the fundamental assumption of centrality fails. Almost all of these applications are characterized by a set of interconnected subsystems, which have access to different information. There is usually a common objective to be achieved by some type of coordination among the local stations.

One of the most recent applications in control is coordination of a set of autonomous vehicles. This includes many different scenarios, from underwater autonomous vehicles to robotic systems in large industrial complexes. These systems are currently very far from full decentralization. Even though there might be a hierarchical or a multi-level structure, all the information is usually shared among all the stations and the strategies are designed and implemented using centralized methods.

Some inspiration for our research came from two current projects in the School of Engineering and Applied Science at UCLA. They represent two forms of coordinated autonomous vehicle systems, namely ultra-light Unmanned Air Vehicles (UAVs) in formation and Commercial Heavy Vehicles (CHVs) in platoons. In this section, we provide short descriptions of these two applications.

2.5.1 Formation Flight

Formation flight is a situation where different aircraft have to coordinate in order to achieve a common goal. Ultra-light Solar Powered UAVs, which are going to be flown in formation are being developed and tested at UCLA. Such a formation could be used as a platform for communication stations, environmental studies, surveillance applications, etc. The basic idea for the formation flight is that each aircraft can take advantage of the up-wash coming off the aircraft in front to reduce its workload and hence save a considerable amount of energy that makes it possible for the aircraft to stay aloft permanently, while powered only by the sun.

In this scenario, each aircraft acts as a local station. They are at the same altitude and their wing tips line up. They are separated longitudinally by one wingspan (43.3 ft). There are going to be five planes in formation and each plane has 12 states. Therefore the global state vector is composed of 5 sets of 12, ordered from the leading plane to the trailing plane. The states for each aircraft are as follows:

- x : Inertial Longitudinal Position (ft)
- y : Inertial Lateral Position (ft)
- z : Inertial Horizontal Position (ft)
- ϕ : Body Axis Roll Angle (rad)
- θ : Body Axis Pitch Angle (rad)
- ψ : Body Axis Yaw Angle (rad)
- u : Body Axis Velocity (x direction) (ft/s)
- v : Body Axis Velocity (y direction) (ft/s)
- w : Body Axis Velocity (z direction) (ft/s)
- p : Body Axis Roll Rate (rad/s)
- q : Body Axis Pitch Rate (rad/s)

- r : Body Axis Yaw Rate (rad/s)

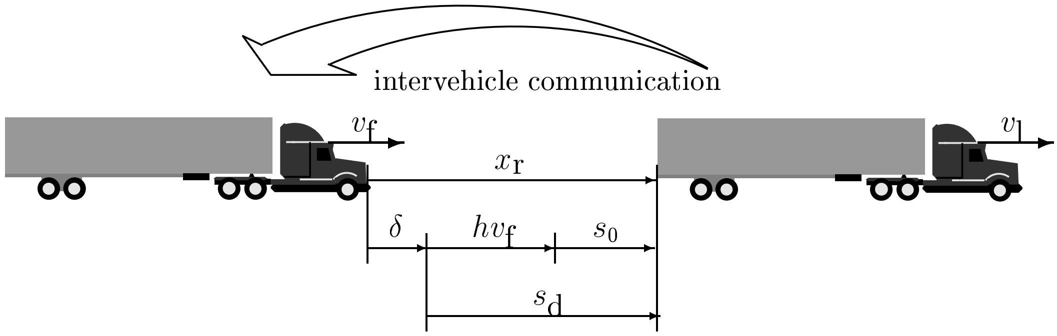
Each aircraft has five controls:

- Right Aileron Deflection (deg)
- Left Aileron Deflection (deg)
- Right Tail Deflection (deg)
- Left Tail Deflection (deg)
- Thrust (lb)

Each aircraft also has its own set of sensors, such as Inertial Measurement Units (IMUs), Global Positioning System (GPS) receivers, air data sensors, etc. This clearly shows why information is indeed decentralized. The goal is to design decentralized strategies to keep the aircraft in formation. Thus, we need to investigate what information should be communicated among the planes. For example, by measuring the phase difference between the carrier waves of the GPS signals, received by different aircraft, accurate relative position and velocity estimates can be obtained. This requires the communication of GPS measurements to neighboring aircraft. In general, we need to find the best information pattern for the aircraft formation so that decentralized control strategies can be designed for each aircraft in order to achieve a specified level of performance.

2.5.2 Vehicle Platoons

Another situation where autonomous vehicles act in a coordinated fashion is in highway automation applications, where tightly spaced vehicle group formations or platoons are formed. Platooning provides significantly higher traffic throughput while also improving fuel economy. These are more important for Commercial Heavy Vehicles, which usually travel on well-established commercial routes. Figure (2.2) shows a typical scenario with the parameters involved.



s_0 : minimum distance between vehicles

h : time headway (for speed-dependent spacing)

x_{r} : vehicle separation

$s_{\mathrm{d}} = s_0 + hv_{\mathrm{f}}$: desired vehicle separation

v_{l} : velocity of leading vehicle

v_{f} : velocity of following vehicle

$v_{\mathrm{r}} = v_{\mathrm{l}} - v_{\mathrm{f}}$: relative vehicle velocity

$\delta = x_{\mathrm{r}} - s_{\mathrm{d}}$: separation error

Figure 2.2: Parameters of a truck platoon.

Every platoon is indeed a decentralized system with each truck acting as a local station. The objective is to regulate the relative velocity and the separation error, which could be expressed as regulating $v_r + k\delta$, where k is a positive design constant. Even though highly nonlinear and detailed models for the trucks are available [86], for the purpose of control design, a simple first order linear model might be used as an approximated longitudinal truck model relating the vehicle speed to the fuel command input.

The operation could be autonomous in the sense that each vehicle measures only its own velocity and the relative velocity and the separation of the preceding vehicle with some uncertainty. In other words, there is no information transmission among the trucks. In an alternative scenario, however, intervehicle communication would be allowed. The specific form of the information pattern is again of vital importance in the performance of the platoon. For example, it is shown in [86] that *string stability* cannot be achieved under the autonomous operation when the time headway is zero, that is, when the desired spacing is independent of vehicle speed. Furthermore, in [22], different types of transmitted information, such as the leader's current velocity and the leader's desired velocity, have been compared based on their effect on the string stability of the platoon. This shows the necessity of a systematic method for evaluating a piece of information based on its effect on the performance of the decentralized system.

2.6 Application Examples in Communications

Decentralized systems have also found applications in communication systems and networks. In this section, we briefly review two such applications.

2.6.1 Congestion Control in ATM Networks

Asynchronous Transfer Mode (ATM) protocol has been proposed as a standard for the next generation Broadband Integrated Services Digital Networks, which should be capable of carrying different types of traffic such as voice, video, and data at very high speeds, while providing the desired Quality of Service for each of these traffic types.

ATM networks are packet switched networks, where the data is packetized into equal length cells. They are also connection-oriented in the sense that there is a signaling phase before any call connection, where the users would set up a Virtual Circuit (VC) and would inform all the intermediate nodes of their traffic characteristics, service class, and required Quality of Service (QoS).

The ATM networks provide four main service categories:

- Constant Bit Rate (CBR): Constant cell rate, emulates circuit switching, no rate control is necessary, e.g., voice.
- Variable Bit Rate (VBR): Variable cell rate possible, no rate control is implemented. Real time VBR and non-real time VBR are considered as two subclasses.
- Available Bit Rate (ABR): Mainly for bursty traffic, e.g., file transfers, No strict QoS requirement are to be imposed by the user. However, delay and losses are to be minimized, and rate control is therefore essential.
- Unspecified Bit Rate (UBR): No QoS guarantees, no rate control, the cells are simply dropped upon congestion.

Two main approaches have been proposed for congestion control for the ABR service category in ATM networks:

- Credit-based Congestion Control: Regulate the *number* of incoming cells using per-link, per-VC window flow control.
- Rate-based Congestion Control: Regulate the incoming cell *rate*, based on the congestion status of the network.

Even though credit-based approaches can guarantee zero cell loss and usually have lower ramp-up times, they need more accurate delay estimates. More importantly, per-VC queueing is essential in credit-based approaches. This results in more complicated switches. These reasons along with the pressure from the switch vendors finally led the ATM Forum to adopt the rate-based approach as the standard scheme for congestion control in ATM networks.

The basic idea is to keep the queue lengths at all nodes or at some bottleneck nodes of the network close to some threshold values and therefore avoid both congestion and under-utilization of the network. Control algorithms should be designed to obtain the explicit rates for the sources, based on the available information, such as the queue length measurements. So the source nodes, each of which can be regarded as a local station, should coordinate in order to achieve a common objective.

Recently it was shown in [1] how a team problem and also a non-cooperative game problem can be formulated to obtain the explicit rates for all the sources in the network.

A continuous fluid approximation is assumed for the traffic flow. It is argued that in today's high speed networks the buffer sizes can be very large such that the error in replacing the number of packets with a real number instead of an integer is small relative to the buffer size. Also it is assumed that there is a single bottleneck link in the network, based on which all the performance measures are defined.

Let $x_q(t)$ be the queue length at a bottleneck link. $s(t)$ is the total available bandwidth at that link. This available rate is assumed to be arbitrary but perfectly known by all the users. Also each user (source) $m \in \mathcal{M} \triangleq \{1, 2, \ldots, M\}$ has access to a given fixed portion of the bandwidth $a_m s(t)$ where $\sum_{m=1}^{M} a_m = 1$. Let $r_m(t)$ be the controlled explicit rate of source m and let $u_m(t) \triangleq r_m(t) - a_m s(t)$ be its shifted version. Also let $x(t) \triangleq x_q(t) - x_0$ be the deviation of the queue length from its desired target value x_0. The queue dynamics can then be written as:

$$\frac{dx}{dt} = \sum_{m=1}^{M}(r_m - a_m s) = \sum_{m=1}^{M} u_m \tag{2.21}$$

Note that a linearized dynamics is assumed for the queue model where the end point effects are neglected. This is justified using the fact that the controlled queue length will be close to its threshold value. Also note that the delays are not taken into account.

Now, the objective is to obtain the explicit rates for all the sources, that is, the control policies $\mu_m \in \mathcal{U}_m$, where $u_m(t) = \mu_m\left(t, x_{[0,t]}\right), \ \ m \in \mathcal{M}, \ \ t \in [0, \infty)$. It is shown that both in team formulation and in non-cooperative game formulation, the optimal policies $u_m^*(t)$ linearly depend only on the current value of $x(t)$. This comes from the fact that all the sources are assumed to have access to the same information, that is, the measurement of the queue length at the bottleneck link.

2.6.2 Power Control in Cellular Wireless Systems

Another application in communications, where decentralized algorithms, in uncertain environments, are to be designed, is power control in cellular radio systems.

In a cellular system the area under coverage is divided into cells and each cell has its own base station. All users communicate with their assigned base stations through a single hop. This is in contrast to *ad hoc* wireless networks where there is no fixed infrastructure and multi-hop communication is prevalent.

Each mobile user acts as a local station. On every single channel, the cellular network can then be considered as a collection of these local stations. These co-channel mobile stations are interacting through the interference that they are causing each other.

The main idea is to control the transmit power levels of the users and the base stations in order to maintain an acceptable level of quality of service for every user, while eliminating unnecessary interference to other users in the network.

Different objectives and approaches have been perceived for power control and different algorithms have been naturally obtained. The major objective in Direct Sequence Code Division Multiple Access (CDMA) systems is to mitigate the multiple access interference and therefore the near-far effect, whereas in Time/Frequency Division Multiple Access (TDMA-FDMA) systems the objective is mostly to control the co-channel interference. Power control will also minimize the power consumption for the users and hence prolong their battery life.

We decided to focus more on this application in our research. Specifically, we anlayzed various approaches for power control design. We showed how power control can be regarded as a decentralized regulator problem and then obtained some global stability results. We also designed and simulated a decentralized integrated predictive power control and dynamic channel assignment scheme.

We will explain the power control problem in full detail in Part IV of this thesis.

2.7 Summary

The purpose of this chapter was to review some basic concepts in team theory and decentralized stochastic control theory. We also introduced different information patterns and explained their characteristics and their effects on the control design.

We conclude that, in general, there are four basic factors, which would generate non-classical information patterns:

- *Limited memory*, i.e., when some stations do not have perfect recall.

- *Constraints on information transmission*, i.e., when it is very costly or maybe even impossible to communicate some information among the stations.

- *Data latency*, i.e., when there is a delay in transmitting the information among the stations.

- *Transmission noise*, i.e., when the information is being corrupted by noise while in transmission.

As soon as the information pattern in a decentralized control system becomes non-classical, finding the optimal strategies becomes very difficult. One basic difficulty comes from the fact that the cost is no longer a convex function of the strategies and there is no guarantee that the optimal strategies are unique. Also, linear strategies would not be optimal anymore.

We will elaborate more on these problems when we discuss a classical example and its reformulations later in Chapter 5.

We also reviewed two application examples in control along with two application examples in communications. The objective was to show, at least in a general sense, how such applications can fit into the abstract platform of a decentralized system, where multiple stations with local information are to be controlled or coordinated to achieve some common objectives.

Part II

Information Transmission in Decentralized Systems

CHAPTER 3

Information Value

3.1 Introduction

The best achievable performance in a decentralized system highly depends on the prevailing information pattern in that system, i.e., *who knows what and when.* Therefore, in order to design decentralized controllers, we should also be looking at different alternatives for distributing the information among the control stations. In the previous chapter, we discussed different information patterns and their characteristics. We saw that when one station does not have access to the previous actions of those other stations that have affected its current information, optimal strategies would be very difficult to find and, in general, are usually unknown.

Another important factor in designing information patterns is the ability to evaluate a piece of information based on how it affects the performance criterion. In other words, we somehow need to know how valuable and how critical a piece of information is, in order to study the feasibility of installing a sensor to get that information or a transmitter to communicate it with other stations.

Even though there has been some effort towards finding such a measure for many years, the success is far from convincing. One reason is the fact that evaluating information, based on its effect on the performance, depends heavily on the specific problem at hand. The only general property for such a measure is that free additional information should never do harm, even though it might well be useless [85]. But still there is no general systematic method to evaluate a piece of information in a specific problem with a given performance index.

In this chapter, we will discuss different approaches to finding such a measure for information. First, we will discuss the entropy approach to estimation and control, where relations have been established between information theory and estimation and control theories. Although one of the main objectives behind this approach was to explain the dual control effect in adaptive controllers and how the information about the unknown parameters could be passed to the estimator through the control actions, it could also be considered as a platform for information evaluation during the design stage. Then, we describe a proposed measure for the value of information based on how it affects the cost. In Section 3.4, we will consider a classical infinite horizon LQG problem and we will see how the information content of the measurements could be characterized by their corresponding noise covariances. We will also characterize some measurements as being critical, if they directly affect the detectability of the system. In the last section, we will discuss another approach in which the information is again characterized by the noise covariances and is directly incorporated into the cost in order to find the minimum required information.

3.2 Entropy Approach to Estimation and Control

Information theory, developed mainly by Shannon [65], provides a concrete mathematical framework for communication systems. One of the basic notions in information theory is the notion of *entropy*, which is defined for a random variable based on its probability distribution. It is actually a measure of the information available in that random variable based on its average uncertainty. *Mutual in-*

formation is also defined as a measure of information that one random variable contains about another one [16].

Many researchers have attempted to make connections between information theory and estimation theory [12, 24, 45, 82]; information-theoretic criteria for estimation have been established and the connections with the classical Bayesian approach have been shown. There have also been efforts in linking information theory with control theory [64, 74, 83], where alternative formulations for optimal (stochastic) control problems have been developed based on some information-theoretic concepts. The main motivation for this new approach was to deal with dual control problems where the optimal control has two, often conflicting, objectives, namely *probing*, which requires the controller to inject excitation signals into the system in order to generate information about the unknown states or parameters and pass it to the estimator, and *regulation*, which requires that the system be excited as little as possible.

In this section, we will provide an overview of the results in the information-theoretic approach to the estimation and control problems. Since entropy, as a measure of information, is explicitly present in this approach, we try to explore the idea of incorporating the information available to the controllers by using the entropy measure in the cost, which is itself expressed as an entropy measure in this case. Later in Section 5.6, we will also describe a decentralized control problem, formulated as a communications problem, where information-theoretic concepts have been used in order to derive the optimal strategies.

Let X be a random variable. Its *entropy* (also called *differential entropy* for continuous random variables) is defined as follows:

$$H(X) = \int p_X(x) \log\left(\frac{1}{p_X(x)}\right) dx = -\int p_X(x) \log p_X(x) dx = E\left[-\log p_X(x)\right], \tag{3.1}$$

where $p_X(x)$ is the Probability Density Function (pdf) for X. When the "log" is in base 2, the entropy is measured in *bits* and when the base is e, it is measured in *nats*. We will mostly use the latter. Entropy is a measure of uncertainty for a random variable. In fact, for many scalar random variables, there is a one-to-one relationship between the entropy and the variance [45]. The *conditional entropy* of a random variable x, given the value for another random variable z, and their *joint entropy*, are defined respectively as follows:

$$H(X|Z) = -\int\int p_{XZ}(x,z)\log p_{X|Z}(x|z)\,dxdz = E\left[-\log p_{X|Z}(x|z)\right] \quad (3.2)$$

$$H(X,Z) = -\int\int p_{XZ}(x,z)\log p_{XZ}(x,z)dxdz = E\left[-\log p_{XZ}(x,z)\right]. \quad (3.3)$$

The *mutual information* between these two random variables is defined as:

$$\begin{aligned} I(X;Z) &= \int\int p_{XZ}(x,z)\log\frac{p_{XZ}(x,z)}{p_X(x)p_Z(z)}dxdz \\ &= H(X) - H(X|Z) = H(Z) - H(Z|X). \end{aligned} \quad (3.4)$$

It can be shown that conditioning always reduces entropy (i.e., $H(X|Z) \leq H(X)$). Therefore the mutual information is always non-negative, and it is zero if and only if the two random variables are independent. Also we can see that it can actually be interpreted as the reduction in the uncertainty of one random variable caused by knowing the other. The following properties are also easy to prove [16]:

$$H(Z) = H(X) + E\left[\log|J|\right], \quad Z = F(X), \quad J = \det\left[\frac{\partial f_i(x)}{\partial x_j}\right] \quad (3.5)$$

$$H(X,Z) = H(X) + H(Z|X) = H(Z) + H(X|Z) \quad (3.6)$$

$$I(X;Z) = I(Z;X) = H(X) + H(Z) - H(X,Z). \quad (3.7)$$

In the Bayesian approach to the estimation problem, a random vector x is estimated based on an observation vector z, so that the average of a scalar cost functional of the error $C(\tilde{x})$, where $\tilde{x} \triangleq x - \hat{x}(z)$, is minimized. The cost functional is desired to be convex and symmetric about its minimum at $\tilde{x} = 0$.

However, the estimation criterion could also be based on the error probability density function $p(\tilde{x})$, which is desired to be concave and symmetric about its maximum at $\tilde{x} = 0$. Since "log" is a monotonically increasing function, one such criterion could be the *error entropy* $H(\tilde{X}) = E[-\log p(\tilde{x})]$. Since $\tilde{x} = x - \hat{x}(z)$, the determinant of the Jacobian of the transformation from (x, z) to $(\tilde{x}, z)$ is unity and hence from (3.5), we have:

$$H(X, Z) = H(\tilde{X}, Z). \tag{3.8}$$

At the same time, from (3.6) and (3.7), we know:

$$H(X, Z) = H(Z) + H(X|Z) \tag{3.9}$$

$$H(\tilde{X}, Z) = H(\tilde{X}) + H(Z) - I(\tilde{X}; Z). \tag{3.10}$$

Hence:

$$H(\tilde{X}) = H(X|Z) + I(\tilde{X}; Z). \tag{3.11}$$

But $H(X|Z)$ does not depend on the estimation procedure. Therefore, minimizing the error entropy is equivalent to minimizing the mutual information between the error and the observation, which could be considered as another information-theoretic criterion for estimation, i.e.,

$$\min_{\hat{X}} H(\tilde{X}) \Longleftrightarrow \min_{\hat{X}} I(\tilde{X}; Z). \tag{3.12}$$

Also, since the mutual information is always non-negative, the error entropy is always lower bounded by $H(X|Z)$, which is only a function of the statistical relationship between x and z. This lower bound could be achieved when the mutual information between the error and the observation is zero, i.e., when the error is statistically independent of the observation:

$$H(\tilde{X}) = H(X|Z) \Leftrightarrow I(\tilde{X}; Z) = 0. \tag{3.13}$$

This is analogous to the *Orthogonality Principle* in the Bayesian approach. As we know, for a linear Gaussian system, the Kalman Filter minimizes the mean square error and the estimation error is uncorrelated and hence independent (because of the Gaussian assumption) of the observations. Therefore, the Kalman Filter is indeed minimizing the estimation error entropy and the mutual information between the estimation error and the observations. For linear stochastic systems with general noise processes, a lower bound on the achievable mutual information between the estimation error and the observation is derived in [24], and it is shown that under some reachability and observability conditions, zero mutual information between the estimation error and the observation can be achieved by using an affine filter, only if the noises are Gaussian.

So far, we have studied an entropy approach to the estimation problem. We shall now discuss an entropy formulation for an optimal (stochastic) control problem. As we shall see, even though these interesting connections have been established between information theory and control and estimation theories, so far they have not led us to a solution for a general control problem, where separation principle fails. As we said earlier, we think of these results as a possible platform to develop a systematic method for evaluating the information, mainly in a decentralized system, where the information is distributed and some information needs to be communicated among the stations.

The following formulation is a slightly modified version of what has been proposed in [64] for continuous-time systems and later in [74] for discrete-time systems. Given a dynamic system and a performance index, the basic idea is to think of the optimal control design in terms of selecting the optimal policy from a set of admissible policies, where the uncertainty of such a selection is represented by a probability density function.

Consider the following deterministic dynamic system:

$$\dot{x} = f(x, u, t) \ , \quad x(t_0) = x_0, \tag{3.14}$$

where $x(t) \in \Omega_x$ is the n-dimensional state vector. The objective is to design the optimal control $u^* \in \mathcal{U}_{ad}$, where $\mathcal{U}_{ad}$ is the set of admissible policies, in order to minimize the following cost function:

$$J(u) = \int_{t_0}^{t_f} L(x, u, t) dt. \tag{3.15}$$

Let $\upsilon \in \Upsilon$ be a fictitious random variable with which $\mathcal{U}_{ad}$ is parameterized, i.e., the control is chosen based on the value that this random variable takes. In other words, u is now considered as a proper function of υ such that the expectation $E_\upsilon \left[J\left(u(\upsilon) \right) \right]$ is well-defined. Then from the fundamental lemma (2.3), we have:

$$\min_{u(.)} E_\upsilon \left[J\left(u(\upsilon) \right) \right] = E_\upsilon \left[\min_u J(u) \right] = \min_u J(u) \triangleq J_{\min}. \tag{3.16}$$

We shall now formulate $\min_{u(.)} E_\upsilon \left[J\left(u(\upsilon) \right) \right]$ as an entropy minimization problem. Let $p(\upsilon)$ be the corresponding probability density function for υ. Then its entropy can be defined as:

$$H(\upsilon) = - \int_\Upsilon p(\upsilon) \ln p(\upsilon) d\upsilon. \tag{3.17}$$

Following *Jaynes' Maximum Entropy Principle* [44], we select $p(\upsilon)$ in order to maximize the above entropy, subject to:

$$\int_\Upsilon p(\upsilon) d\upsilon = 1, \tag{3.18}$$

$$E_\upsilon \left[J\left(u(\upsilon) \right) \right] = J_{\min}. \tag{3.19}$$

Using the Lagrange multipliers λ_1 and μ, we form the following functional:

$$\begin{aligned} I &= H(\upsilon) - \mu \left(E[J] - J_{\min} \right) - \lambda_1 \left(\int_\Upsilon p(\upsilon) d\upsilon - 1 \right) \\ &= - \int_\Upsilon \left(p \ln p + \mu J p \right) d\upsilon - \lambda_1 \left(\int_\Upsilon p(\upsilon) d\upsilon - 1 \right) + \mu J_{\min}. \end{aligned} \tag{3.20}$$

Maximization with respect to $p(v)$ requires:

$$\frac{\partial I}{\partial p(v)} = \frac{\partial}{\partial p(v)} \left(-p\ln p - \mu J p - \lambda_1 p\right) = -\ln p - 1 - \mu J - \lambda_1 = 0 \tag{3.21}$$

$$\frac{\partial^2 I}{\partial p^2(v)} = -\frac{1}{p} < 0. \tag{3.22}$$

Hence:

$$p(v) = \exp\left(-\lambda - \mu J\left(u(v)\right)\right), \tag{3.23}$$

where $\lambda \triangleq \lambda_1 + 1$ and μ are appropriate positive constants, which can be obtained by substituting back into the constraints. The maximized entropy corresponding to the above density is:

$$H(v) = -\int_{\Upsilon} p(v)\ln p(v) dv = \lambda + \mu E\left[J\left(u(v)\right)\right]. \tag{3.24}$$

Using (3.16), it is clear that minimizing the above entropy is equivalent to minimizing our original cost. Therefore, we can state the following theorem:

Theorem 2. *[64] A necessary and sufficient condition for u^* to minimize $J(u)$ subject to $\dot{x} = f(x,u,t)$, $x(t_0) = x_0$, is that u^* minimizes the entropy $H(v)$, where v is a fictitious random variable that parameterizes the admissible control policies $\mathcal{U}_{ad}$ and $p(v)$, its corresponding probability density function, is selected according to Jaynes' principle to maximize the above entropy.*

The above result can be generalized to stochastic optimal control problems. Consider the following stochastic system:

$$\dot{x}(t) = f(x,u,w,t), \tag{3.25}$$

$$z(t) = g(x,v,t), \tag{3.26}$$

where $x(t_0) = x_0$ is the random initial condition with probability density function (pdf) $p(x_0)$, w is the process noise with pdf $p(w(t))$ and v is the measurement noise with pdf $p(v(t))$. The cost is now given as an average value on all the underlying uncertainties, i.e.,

$$J_s(u) = E_{x_0,w,v}\left[\int_{t_0}^{t_f} L(x,u,t)dt\right]. \tag{3.27}$$

A similar formulation can now be obtained simply by replacing J in the deterministic problem by J_s and following the same procedure, after introducing a new fictitious random variable.

This formulation has an interesting property. Let ϕ denote the unknown state or an unknown parameter in the system, which needs to be estimated, in order to implement the optimal control. We always have:

$$\begin{aligned}
H(v) &= -\int_{\Upsilon} p(v)\ln p(v)dv \\
&= -\int_{\Upsilon}\int_{\Phi} p(v,\phi)\ln p(v)d\phi dv \\
&= -\int_{\Upsilon}\int_{\Phi} p(v,\phi)\ln \frac{p(v|\phi)p(\phi)}{p(\phi|v)}d\phi dv \\
&= -\int_{\Upsilon}\int_{\Phi} p(v,\phi)\ln p(v|\phi)d\phi dv \\
&\quad -\int_{\Upsilon}\int_{\Phi} p(v,\phi)\ln p(\phi)d\phi dv \\
&\quad +\int_{\Upsilon}\int_{\Phi} p(v,\phi)\ln p(\phi|v)d\phi dv \\
&= H(v|\phi)+H(\phi)-H(\phi|v).
\end{aligned} \tag{3.28}$$

The first term represents the certainty equivalent control, the second term is the entropy corresponding to the unknown itself and the third term represents the information that needs to be transmitted by the controller about the unknown.

3.3 A Definition for the Value of Information

Information-theoretic measures, such as entropy, which were mainly developed for communication systems and information transmissions, evaluate a piece of information only based on its probabilistic characterizations. In other words, they only give a measure of uncertainty without considering the consequences of getting or denying that piece of information. However, in control systems, we are concerned not only with the probabilistic nature of the uncertainties, but also with the impact of these uncertainties on the performance of our system. In other words, in evaluating a piece of information, we should also consider how that piece of information affects our objective. Efforts towards developing such a theory for the *value of information* was originally initiated by Howard in [42]. In this context, the value of information is basically defined as the change in the optimized performance index, when the information is known, relative to the case where the access to that information is denied. The relation between this notion and the entropy measure and how they could both be combined in order to evaluate a piece of information are explained in [66].

Consider a decentralized system. Let z_i be the information available to the i-th station and $\bar{z}_i$ be the information available to all other stations. Also let γ_i be the strategy for the i-th station and $\bar{\gamma}_i$ be the strategies for all other stations. Let ξ denote the underlying uncertainties in the system. The objective is to minimize the following cost:

$$J = E_{z_i,\bar{z}_i,\xi}\left[L\left(\gamma_i(z_i), \bar{\gamma}_i(\bar{z}_i), \xi\right)\right]. \tag{3.29}$$

We want to find the value of the information available at the i-th station, that is, the value of z_i based on how it affects our cost.

When the i-th station has access to z_i, the minimum cost that could be achieved is the following:

$$J_1 = \min_{\gamma_i \in \Gamma_i, \bar{\gamma}_i \in \bar{\Gamma}_i} E_{z_i, \bar{z}_i, \xi} \left[L\left(\gamma_i(z_i), \bar{\gamma}_i(\bar{z}_i), \xi \right) \right]. \tag{3.30}$$

On the other hand, when the access of the i-th station to z_i is denied, the minimum achievable cost would be:

$$J_2 = \min_{u_i \in U_i, \bar{\gamma}_i \in \bar{\Gamma}_i} E_{\bar{z}_i, \xi} \left[L\left(u_i, \bar{\gamma}_i(\bar{z}_i), \xi \right) \right]. \tag{3.31}$$

Note that we have not eliminated the i-th station, but we have just denied its access to its information. Therefore, its best strategy, which should be independent of z_i, is clearly a constant strategy that yields the minimum cost J_2.

The *value of the information* z_i is now defined as the difference between the corresponding costs in these two cases, i.e.,

$$\text{value of } z_i \stackrel{\triangle}{=} J_2 - J_1. \tag{3.32}$$

Note that the above quantity is always positive. This is also clear from the fact that knowing z_i cannot increase the minimum cost, even though it may not decrease it that much either.

Finding the value of information using this definition requires that we be able to find the optimal strategies in both cases, when we have access to the information and when we do not. In other words, this approach somehow addresses the question of having or not having a sensor, when we know how to incorporate the sensor's output in our optimal strategy.

3.4 Cost versus Information in the Classical LQG Problem

So far, we have discussed two different approaches to evaluating information in a control system, namely the information-theoretic approach and the cost-dependent approach. However, none of these two approaches, at their current stages, have led us to a systematic method for information pattern design in decentralized systems. Another possible approach is to characterize the information content of a measurement simply by its corresponding noise covariance, which is again a measure of uncertainty.

In this section, we will first investigate the behavior of the optimum cost in the centralized LQG problems as the measurement noise covariances change. By looking at an example, we will see that in this case, these noise covariances actually provide us with a good measure for the value of an observation, i.e., how much should be spent on installing a specific sensor. As we shall see, the concept of *detectability* is crucial in identifying the critical measurements in a centralized system. Next, we will see how this approach could be generalized for decentralized systems to obtain a value for a piece of information, which is being communicated among the stations.

We now state the centralized Linear Quadratic Gaussian (LQG) problem [11]. Consider the following linear stochastic system:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \tag{3.33}$$

$$z(t) = Hx(t) + v(t), \tag{3.34}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the control vector and $z(t) \in \mathcal{R}^r$ is the measurement vector. $w(t)$ and $v(t)$ are the process noise and the measurement noise respectively, which are assumed to be independent white Gaussian processes with the following properties:

$$E\left[w(t)\right] = 0, \quad E\left[v(t)\right] = 0, \tag{3.35}$$

$$E\left[\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w^T(\tau) & v^T(\tau) \end{bmatrix}\right] = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-\tau), \tag{3.36}$$

where $W = W^T \geq 0$ and $V = V^T > 0$.

The objective is to minimize the following quadratic cost:

$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt\right]. \tag{3.37}$$

We know that under stabilizability and detectability conditions, the optimal control is the following:

$$u(t) = -K\hat{x}(t), \quad K = R^{-1}B^T\Pi, \tag{3.38}$$

where Π is the unique symmetric positive definite solution of the control algebraic Riccati equation:

$$-\Pi A - A^T\Pi + \Pi BR^{-1}B^T\Pi - Q = 0, \tag{3.39}$$

and the state estimate $\hat{x}$ is obtained from the following Kalman filter:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left(z(t) - H\hat{x}(t)\right), \tag{3.40}$$

where $L = PH^TV^{-1}$ and P is the unique positive definite solution of the filter algebraic Riccati equation:

$$AP + PA^T - PH^TV^{-1}HP + W = 0. \tag{3.41}$$

Using the above control, the optimum cost will be obtained as follows:

$$J^* = tr\left(\Pi W + K^TRKP\right) = tr\left(QP + LVL^T\Pi\right). \tag{3.42}$$

As we can see, the measurement noise covariances (elements of V) change the gain in the Kalman filter by changing the solution to the filter algebraic Riccati equation. As the noise covariance for a measurement increases, the corresponding gain for that measurement decreases, i.e., the Kalman filter puts less emphasis on that measurement.

Let us assume that the measurement noises are uncorrelated, i.e., V is diagonal. If we now plot the optimum cost J^* versus the noise variance corresponding to the i-th measurement, we will get a monotonically increasing function. This plot is very helpful in evaluating a measurement based on its effect on our optimum cost. We will see that for some measurements, the optimum cost will go to infinity when the variance increases to infinity. These are the measurements, which we consider as critical in the sense that they render the system *undetectable* if they are eliminated. However, for other measurements, we can decide how much we should be spending on their corresponding sensors based on how much they affect our optimum cost.

Consider the following system as an example:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) + w(t) \tag{3.43}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}. \tag{3.44}$$

The system matrix has three distinct eigenvalues at 1, -2 and 0. It is easy to see that the system is completely controllable. It is also observable if both measurements are incorporated. However, if the second measurement z_2 is eliminated, then the unstable mode 1 will become unobservable, which will make the system undetectable.

Figure (3.1) shows how the optimum cost will change if we change the corresponding noise covariances on the two measurements, when an LQG controller is applied. We can see that the cost goes to infinity when the noise on the second measurement becomes large.

Unfortunately, it is very difficult to generalize this concept to decentralized

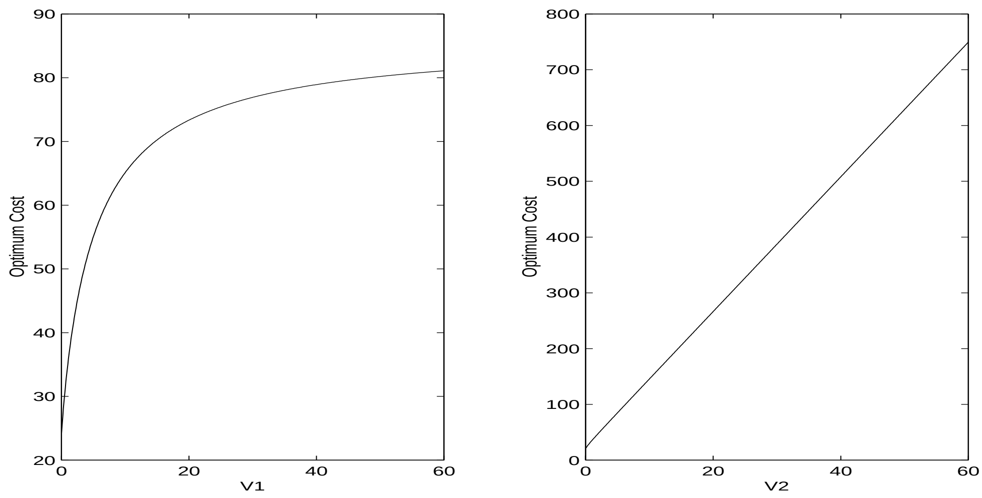


Figure 3.1: Optimum cost behavior versus noise intensity

systems. As we saw in the first chapter, the main challenge in decentralized systems is that we do not know how to find the optimal controls as soon as the information pattern becomes non-classical and it is not in a partially nested form. Even though we may be able to identify the piece of information that needs to be obtained or transmitted in order to have at least a partially nested pattern, this generally does not imply that denying that piece of information would yield an infinite cost. For the same reason, there is no known solution for the LQG problem for decentralized systems with general non-classical information patterns. In [70], a decentralized formulation for the LQG problem is presented. But even though the state estimate is obtained by combining the estimates of the local estimators, which use the local information, the optimal control is not decentralized in the sense that each station is allowed to use all the sensor data.

However, we may use a *suboptimal* approach in a decentralized LQG problem and try to evaluate a piece of information that is being communicated among the stations. For simplicity, consider a linear system with only two stations:

$$\dot{x}(t) = Ax(t) + B_1u^1(t) + B_2u^2(t) + w(t) \tag{3.45}$$

$$z^1(t) = H_1x(t) + v_1(t) \tag{3.46}$$

$$z^2(t) = H_2x(t) + v_2(t). \tag{3.47}$$

The original objective is to find $u^1 = u^1(z^1)$ and $u^2 = u^2(z^2)$in order to minimize the following cost:

$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(x^T(t)Qx(t) + {u^1}^T(t)R_1u^1(t) + {u^2}^T(t)R_2u^2(t)\right) dt\right]. \tag{3.48}$$

Now assume that the two stations start communicating through an additive white Gaussian noise channel. So the new measurements are:

$$z^1(t) \triangleq \begin{bmatrix} z_1^1(t) \\ z_2^1(t) \end{bmatrix} = \begin{bmatrix} H_1x(t) + v_1(t) \\ H_2x(t) + v_2(t) + v_{21}(t) \end{bmatrix} \tag{3.49}$$

$$z^2(t) \triangleq \begin{bmatrix} z_1^2(t) \\ z_2^2(t) \end{bmatrix} = \begin{bmatrix} H_1x(t) + v_1(t) + v_{12}(t) \\ H_2x(t) + v_2(t) \end{bmatrix}, \tag{3.50}$$

where $v_{12}(t)$ and $v_{21}(t)$ are independent transmission noises, which are also assumed to be independent of other underlying uncertainties in the system.

The optimal decentralized solution would still be a solution in which u^1 is only a function of z^1 and u^2 is only a function of z^2. Since we currently do not know such an optimal solution, we use a suboptimal approach, where we consider the system as two centralized systems. In other words, we solve two centralized LQG problems, namely for the first station:

$$\min\left(J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(x^T(t)Qx(t) + {u_1^1}^T(t)R_1u_1^1(t) + {u_2^1}^T(t)R_2u_2^1(t)\right) dt\right]\right) \Rightarrow$$

$$\begin{bmatrix} u_1^1(t) \\ u_2^1(t) \end{bmatrix} = \begin{bmatrix} -R_1^{-1}B_1^T\Pi\hat{x}^1(t) \\ -R_2^{-1}B_2^T\Pi\hat{x}^1(t) \end{bmatrix}, \tag{3.51}$$

where $\hat{x}^1$ is the state estimate in the first station:

$$\dot{\hat{x}}^1(t) = A\hat{x}^1(t) + B_1u_1^1(t) + B_2u_2^1(t) + P_1H^T\left(V^1\right)^{-1}\left(z^1(t) - H\hat{x}^1(t)\right) \tag{3.52}$$

$$AP_1 + P_1A^T - P_1H^T\left(V^1\right)^{-1}HP_1 + W = 0 \tag{3.53}$$

$$H \triangleq \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad V^1 \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & V_2 + V_{21} \end{bmatrix}, \tag{3.54}$$

and for the second station:

$$\min\left(J = \lim_{T\to\infty}\frac{1}{T}E\left[\int_0^T\left(x^T(t)Qx(t) + u_1^{2^T}(t)R_1u_1^2(t) + u_2^{2^T}(t)R_2u_2^2(t)\right)dt\right]\right) \Rightarrow$$

$$\begin{bmatrix} u_1^2(t) \\ u_2^2(t) \end{bmatrix} = \begin{bmatrix} -R_1^{-1}B_1^T\Pi\hat{x}^2(t) \\ -R_2^{-1}B_2^T\Pi\hat{x}^2(t) \end{bmatrix}, \tag{3.55}$$

where $\hat{x}^2$ is the state estimate in the second station:

$$\dot{\hat{x}}^2(t) = A\hat{x}^2(t) + B_1u_1^2(t) + B_2u_2^2(t) + P_2H^T\left(V^2\right)^{-1}\left(z^2(t) - H\hat{x}^2(t)\right) \tag{3.56}$$

$$AP_2 + P_2A^T - P_2H^T\left(V^2\right)^{-1}HP_2 + W = 0 \tag{3.57}$$

$$H \triangleq \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad V^2 \triangleq \begin{bmatrix} V_1 + V_{12} & 0 \\ 0 & V_2 \end{bmatrix}. \tag{3.58}$$

Note that only the measurement noise characteristics and the filter are different in the two problems. We now apply u_1^1 and u_2^2 as the suboptimal controls to the system and calculate the corresponding cost:

$$J_s^* = \lim_{T\to\infty}\frac{1}{T}E\left[\int_0^T\left(x^T(t)Qx(t) + u_1^{1^T}(t)R_1u_1^1(t) + u_2^{2^T}(t)R_2u_2^2(t)\right)dt\right], \tag{3.59}$$

Then, we investigate the behavior of this cost versus the intensity of the transmission noises $v_{12}(t)$ and $v_{21}(t)$ using the same approach as for the centralized case.

There is no guarantee that the decentralized algorithms, obtained through this sub-optimal approach, could preserve the desirable properties of a centralized LQG controller. In fact, surprisingly enough, these sub-optimal algorithms may even fail to stabilize the global system, even when the stations are communicating all their measurements. We will fully investigate this problem in Chapter 4.

3.5 Incorporating Information in a Quadratic Cost

In the previous section, we saw how an observation could be evaluated simply by looking at the cost behavior versus the observation noise covariance as a measure of uncertainty. In this section, we will use this idea to incorporate a measure for the information cost in the original quadratic cost. In this approach, which was originally proposed in [71], we will first use the results from the LQG problem to minimize the cost with respect to the control. Then we will have a deterministic optimization problem, where another cost needs to be minimized in order to find the minimum necessary information.

Consider the following linear stochastic system:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \tag{3.60}$$

$$z(t) = Hx(t) + v(t), \tag{3.61}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the control vector, $z(t) \in \mathcal{R}^r$ is the measurement vector and $w(t)$ and $v(t)$ are again the process noise and the measurement noise respectively, which are assumed to be independent white Gaussian processes with the following properties:

$$E\left[w(t)\right] = 0, \quad E\left[v(t)\right] = 0 \tag{3.62}$$

$$E\left[\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}\begin{bmatrix} w^T(\tau) & v^T(\tau) \end{bmatrix}\right] = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}\delta(t-\tau), \tag{3.63}$$

where $W = W^T \geq 0$ and $V = V^T > 0$.

The initial condition $x(t_0)$ is assumed to be Gaussian with the mean $\bar{x}_0$ and the covariance P_0. It is also independent of $w(t)$ and $v(t)$ for all $t \in [t_0, t_f]$.

Let the noise intensity matrix V have the following form:

$$V = \bar{V}\Sigma^{-1}, \tag{3.64}$$

where Σ is a diagonal matrix:

$$\Sigma = diag\,(\sigma_{11}, \ldots, \sigma_{rr})\,, \qquad 0 \leq \sigma_{ii} \leq 1, \quad i = 1, \ldots, r. \tag{3.65}$$

Let $\bar{V}$ also be a diagonal matrix, where the i-th diagonal element $\bar{v}_{ii}$ denotes the lowest possible noise intensity for the i-th measurement. Thus, V will also be a diagonal matrix with its diagonal elements in the following form:

$$v_{ii} = \frac{\bar{v}_{ii}}{\sigma_{ii}}, \qquad 0 \leq \sigma_{ii} \leq 1, \quad i = 1, \ldots, r. \tag{3.66}$$

The basic idea is to include a weighted factor of σ_{ii}'s in the cost and try to minimize with respect to them. When $\sigma_{ii} = 1$, we have $v_{ii} = \bar{v}_{ii}$, which represents the most precise observation that we can make. On the other hand, $\sigma_{ii} = 0$ implies that the i-th observation should be disregarded. The intermediate values for σ_{ii} could be interpreted as reducing the sample rate for the i-th measurement or simply as using a less accurate sensor for that measurement.

One way is to construct the cost as the following:

$$J = E\left[\int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t) + q^T\Sigma(t)q\right) dt\right], \tag{3.67}$$

where $q \in \mathcal{R}^r$ is a constant weighting factor, which could be considered as a design parameter. Using the results from the finite horizon LQG problem [11], we first minimize with respect to u:

$$\begin{aligned} u(t) &= K(t)\hat{x}(t) = -R^{-1}B^T\Pi(t)\hat{x}(t) && (3.68) \\ \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + P(t)H^T\Sigma(t)\bar{V}^{-1}\left(z(t) - H\hat{x}(t)\right), && (3.69) \end{aligned}$$

where $\Pi(t)$ and $P(t)$ are the solutions of the following Riccati differential equations:

$$\dot{\Pi}(t) = -\Pi(t)A - A^T\Pi(t) + \Pi(t)BR^{-1}B^T\Pi(t) - Q, \qquad \Pi(t_f) = 0 \tag{3.70}$$

$$\dot{P}(t) = AP(t) + P(t)A^T - P(t)H^T\Sigma(t)\bar{V}^{-1}HP(t) + W, \qquad P(t_0) = P_0. \tag{3.71}$$

This control yields the following cost:

$$\min_u J = tr\left\{\Pi(t_0)P_0 + \int_{t_0}^{t_f}\left(\Pi(t)W + K(t)^TRK(t)P(t)\right)dt\right\} + \int_{t_0}^{t_f} q^T\Sigma(t)q dt. \tag{3.72}$$

We now need to minimize the above cost with respect to the elements of Σ subject to the Riccati differential equation for P, i.e., (3.71), which actually forms a dynamic constraint.

This is a deterministic optimization problem, where we can use the Hamiltonian approach. However, since the dynamic constraint is in the matrix form, we have to use a matrix version of Pontryagin's minimum principle [3]. Based on (3.72), we define the variational Hamiltonian as:

$$\begin{aligned} \mathcal{H} &= tr\left\{\Pi W + K^TRKP + \Sigma qq^T\right\} + \sum_{i=1}^{n}\sum_{j=1}^{n}\dot{p}_{ij}\lambda_{ij} \\ &= tr\left\{\Pi W + K^TRKP + \Sigma qq^T + \dot{P}\Lambda^T\right\} \\ &= tr\left\{\Pi W + K^TRKP + \Sigma qq^T + \left(AP + PA^T - PH^T\Sigma\bar{V}^{-1}HP + W\right)\Lambda^T\right\} \end{aligned} \tag{3.73}$$

The multiplier functions matrix $\Lambda(t)$ satisfies the following:

$$\dot{\Lambda} = -\frac{\partial \mathcal{H}}{\partial P}, \qquad \Lambda\left(t_f\right) = 0. \tag{3.74}$$

Hence[1]:

$$\dot{\Lambda} = -\Lambda\left(A - PH^T\bar{V}^{-1}\Sigma H\right) - \left(A^T - H^T\bar{V}^{-1}\Sigma HP\right)\Lambda^T - K^TRK, \quad \Lambda\left(t_f\right) = 0, \tag{3.75}$$

which also shows that Λ is indeed symmetric, i.e., $\Lambda = \Lambda^T$.

The optimal Σ has to satisfy:

$$\frac{\partial \mathcal{H}}{\partial \Sigma} = 0. \tag{3.76}$$

However, since $\mathcal{H}$ is linear in Σ, we can not determine Σ in terms of P and Λ from the above necessary condition. In other words, we have a *singular surface* in this optimization problem, and we do not know whether it is indeed minimizing or not.

To investigate the nature of the solution further and for simplicity, we continue with the scalar case:

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) + w(t) && (3.77)\\ z(t) &= hx(t) + v(t). && (3.78) \end{aligned}$$

The Hamiltonian is:

$$\mathcal{H} = \Pi W + k^2Rp + \sigma q^2 + \lambda\left(2ap - \sigma p^2 d + W\right), \quad d \triangleq h^2\bar{V}^{-1}, \quad k \triangleq R^{-1}b\Pi, \tag{3.79}$$

where Π and p are again solutions to the control and filter Riccati differential equations respectively. Then the derivative of $\mathcal{H}$ with respect to σ is:

$$\mathcal{H}_\sigma \triangleq \frac{\partial \mathcal{H}}{\partial \sigma} = q^2 - \lambda p^2 d. \tag{3.80}$$

[1]we have used: $\frac{\partial}{\partial X} tr\left\{AX\right\} = A^T$ and $tr\left\{AB\right\} = tr\left\{BA\right\}$

From Pontryagin's minimum principle the Hamiltonian should now be minimized, and therefore we have:

$$\sigma^* = \begin{cases} 0 & if \quad \mathcal{H}_\sigma > 0 \\ 1 & if \quad \mathcal{H}_\sigma < 0, \\ 0 < \sigma_s < 1 & if \quad \mathcal{H}_\sigma = 0 \end{cases} \tag{3.81}$$

where λ and p should be obtained from the following Two Point Boundary Value Problem (TPBVP):

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial p} = -2\lambda(a - \sigma p d) - k^2 R, \quad \lambda(t_f) = 0 \tag{3.82}$$

$$\dot{p} = 2ap - \sigma p^2 d + W, \quad p(t_0) = p_0, \tag{3.83}$$

and σ_s is the value for σ on the singular arc, which should be in $[0, 1]$. Let p_s denote the corresponding value for the error variance p on the singular arc. The singular arc will be characterized by setting $\mathcal{H}_\sigma$ and its time derivatives equal to zero, i.e.,:

$$\mathcal{H}_\sigma = q^2 - \lambda p_s^2 d = 0, \tag{3.84}$$

and:

$$\frac{d}{dt}(H_\sigma) = -2\lambda a p_s^2 d - 2\lambda p_s d W + k^2 R p_s^2 d = 0. \tag{3.85}$$

Substituting for λ from (3.84), we will see that the error variance on the singular arc is, in fact, the non-negative root of the following equation:

$$dk^2 R p_s^3 - 2q^2 a p_s - 2q^2 W = 0. \tag{3.86}$$

We get the value of σ_s by setting the second time derivative of $\mathcal{H}_\sigma$ equal to zero:

$$\frac{d^2}{dt^2}(H_\sigma) = 0 \quad \Longrightarrow$$

$$\sigma_s = \frac{4a^2 q^2 p_s^2 - 4Waq^2 p_s - 3W^2 q^2 - kdb\left(-2a\Pi + b^2 \Pi^2 R^{-1} - Q\right) p_s^4}{2aq^2 d p_s^3 - 3Wq^2 d p_s^2}. \tag{3.87}$$

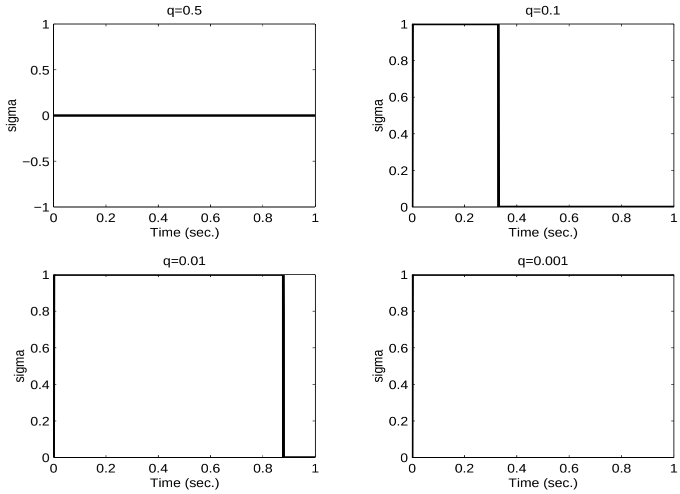


Figure 3.2: A sample behavior of σ

Note that for the finite horizon case, k and hence p_s are changing with time, and we will have a singular arc only when p_s is non-negative and the corresponding σ_s is a value in $[0, 1]$.

At this stage, we do not know how σ will generally behave. We have simulated a simple scalar system with different values for the parameters. Almost in all cases, σ did not seem to be on the singular arc at any time. Figure (3.2) shows how σ behaves for different values of the weighting q in a simple example with the following parameters:

$$\begin{aligned} a &= -1, \qquad b = 1, \qquad h = 1 \\ \bar{V} &= 1, \qquad W = 1, \qquad p_0 = 1 \\ Q &= 1, \qquad R = 1, \\ t_0 &= 0, \qquad t_f = 1, \end{aligned} \tag{3.88}$$

Figure (3.3) shows the corresponding behavior of the error variance p and its value on the singular arc p_s. Note that even though they cross each other and an acceptable value for σ_s can indeed be obtained, the singular arc does not seem

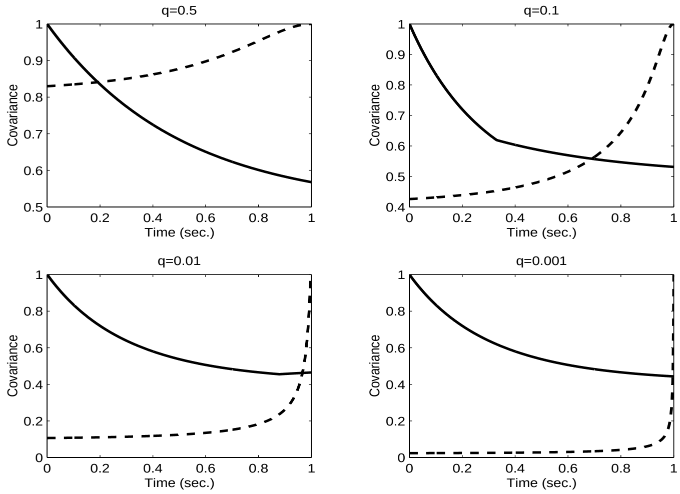


Figure 3.3: Error variance (p: solid) and its corresponding value on the singular arc (p_s: dash)

to be minimizing and σ switches at most once from 1 to 0. This implies that, depending on how expensive the measurement is (determined by the value of q), it should be either taken with its full precision or thrown away.

3.6 Summary

In this chapter, we elaborated on one of our basic objectives, which was to develop a systematic method in order to evaluate a piece of information in a decentralized system. First, we discussed an information-theoretic approach to an estimation problem and then we presented a slightly modified version of a known entropy formulation for a general control problem. As we mentioned, even though some interesting results have been obtained from this alternative approach, they have not yet found applications in solving a general optimal control problem. However, since information measures are explicit in this approach, we expected that it would lead to development of an appropriate framework for information analysis in control systems, especially when the information is decentralized.

We then introduced a measure of information, which basically evaluates a piece of information directly from its effect on the optimal cost. As we saw, in order to find the value of information based on such a measure, we would need to know the optimal strategies with and without the access to the information. Unfortunately, this may not always be true. One should modify this measure and somehow incorporate it directly in the cost and try to find the minimum required information.

Next, we proposed the concept of evaluating a piece of information simply by using the covariance, as a measure of uncertainty, and looking at the behavior of the optimum cost as that covariance changes. We also proposed a way to generalize this idea to decentralized systems. However, when some specific measurements in a decentralized system are corrupted with noise, changing their covariance will not alter the information pattern. In other words, dealing with non-classical information patterns precedes information evaluation for designing decentralized control algorithms. This makes it difficult to generalize the simple definitions for the value of information to general decentralized systems. For this reason, we proposed a suboptimal approach, which will be discussed in more detail in Chapter 4.

Finally, we discussed a method to incorporate a cost for the information in the performance index. We showed that the optimization problem may be solved in two stages. The first stage is a stochastic optimization problem, the solution to which is obtained by using the classical LQG results. The second stage is a deterministic optimization problem, where the Hamiltonian approach can be used. We saw that, with our proposed scheme for incorporating information cost, the second stage of optimization turns out to be a singular optimization problem.

One should note, however, that this is only the first step towards information minimization for control. So it may not be the best scheme to such an end and alternative approaches should be explored.

CHAPTER 4

Decentralized LQG Problem with Noisy Communication

4.1 Introduction

So far we have mentioned some of the difficulties that appear in designing decetralized control algorithms. In fact, even in a deterministic case, it may be very hard to design controllers that, at least, achieve global stability, as soon as the system has unstable fixed modes [81]. Incorporating uncertainties along with some optimization criteria can obviously make the problem even more difficult.

In this chapter, we consider a decentralized Linear Quadratic Gaussian (LQG) problem where the stations are allowed to communicate some pieces of information. All stations are assumed to have linear dynamics while all uncertainties are modeled as Gaussian processes. Moreover, each local controller only has access to its own local information, which includes its own measurements and possibly information received through communication with other stations. Such a decentralized nature of information generally induces a non-classical information pattern for this class of problems. Therefore, except for some special structures, where the information pattern is actually a classical pattern [70], the optimal strategies are usually unknown.

Some sub-optimal approaches, however, might be proposed. One such approach is to treat the problem as a collection of separate centralized problems. A motivation for this approach would become clearer if we assume that each station is allowed to communicate all its measurements through low noise communication channels with all the other stations. Even though a huge burden of computation and communication resources may be needed in this scenario, we would expect the controllers to be very close to the optimal stabilizing decentralized controllers.

In the next section, we formulate a simple two-station decentralized LQG problem. In Section 4.3, we discuss the above mentioned sub-optimal approach, where we propose a solution based on two separate centralized problems. In Section 4.4, we investigate the stability properties of our controllers in various scenarios. Namely, we first consider the case where the stations do not communicate at all. Then, we assume that the stations can communicate their state estimates or equivalently their control values. In these scenarios, as we shall see, there is little justification for our approach. But later, we will discuss the case where the stations are allowed to communicate all their measurements. As we mentioned, our approach seems very reasonable for this scenario, at least when the transmission noise intensities are assumed to be small. However, as our main contribution in this chapter, we will show that even in this case, our controllers may fail to stabilize the closed-loop system. This clearly contradicts what we had expected. In another scenario, the stations will be allowed to communicate both their measurements and their control values. We will see how sending the controls will help us achieve at least the closed-loop stability with our controllers. Then, we assume that the stations communicate only their estimation residuals. We will show that when the transmission noise intensities are small, sending estimation residuals would be enough to achieve closed-loop stability in our sub-optimal approach. In Section 4.5, we will mention how similar results may be obtained for discrete-time systems and finally in Section 4.6, we will provide concluding remarks for this chapter.

4.2 Problem Statement

Consider the following decentralized linear system with two stations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u^1(t) + B_2 u^2(t) + w(t) && (4.1) \\ z^1(t) &= H^1 x(t) + v^1(t) && (4.2) \\ z^2(t) &= H^2 x(t) + v^2(t), && (4.3) \end{aligned}$$

where $x(t) \in \mathcal{R}^n$ is the global state vector, $u_1(t) \in \mathcal{R}^{m_1}$ and $z^1(t) \in \mathcal{R}^{r_1}$ are the control and the information vectors for the first station and $u_2(t) \in \mathcal{R}^{m_2}$ and $z^2(t) \in \mathcal{R}^{r_2}$ are the control and the information vectors for the second station. The process noise and the information noise are denoted by $w(t)$, $v^1(t)$ and $v^2(t)$ respectively, which are all assumed to be zero mean white Gaussian with intensity matrices W, V^1 and V^2. They are also assumed to be mutually independent and independent of the initial state. Note that we distinguish between measurement and information, simply because of the fact that the information vector for one station may also include the transmitted measurements of the other station.

The original objective is to find $u^1 = u^1(z^1)$ and $u^2 = u^2(z^2)$ in order to minimize the following cost:

$$J = \lim_{T \to \infty} \frac{1}{T} E \left[\int_0^T \left(x^T(t) Q x(t) + u^{1^T}(t) R_1 u^1(t) + u^{2^T}(t) R_2 u^2(t) \right) dt \right]. \quad (4.4)$$

Since the stations, in general, have access to different information, we have a non-classical information pattern. Moreover, the information pattern is not partially nested. That is, the information available to each station is being affected by the control action of the other station, while there is no way for that station to obtain any information about those control actions. Therefore, in general, we will have a non-convex functional optimization problem, the solutions of which are usually very difficult to obtain.

One possible sub-optimal approach is to solve two separate centralized problems. We will discuss this approach in the following sections. But there are two points that we need to mention now. We will consider different scenarios where the stations communicate different pieces of information. As we shall see, in many cases, we are, in fact, fixing the structure of our controllers only based on the centralized results. Even though this comes naturally out of our lack of knowledge about the structure of the decentralized controllers, it may well be justified for the case where the stations communicate all their measurements through low noise channels. The other point is the way that we model the transmitted information. We would simply model the received information signal as the transmitted signal plus a Gaussian transmission noise. While this model is realistic for analog communication systems, it may not be well justified when digital communication is used. Namely, in digital communication systems, the signal is quantized, coded and sent through the channel. The channel noise may still be assumed to be additive and Gaussian, but sophisticated modulation and coding schemes make it difficult to assume a simple additive Gaussian uncertainty for the received *information signal*. However, if we try to incorporate the quantization effects along with the bit error probability distribution for some *good* coding and modulation schemes in order to model the communication uncertainties, we will end up with models which could still be approximated, to some degree, by simple additive Gaussian models. Moreover, since there are already major difficulties in dealing with decentralized non-classical information patterns, using more complex models for communication uncertainties may not seem very reasonable at this point. Furthermore, we believe the results obtained under such a simplifying assumption would still be helpful in giving us insight towards the real structure of optimal decentralized controllers.

4.3 A Sub-optimal Approach

One possible sub-optimal approach in dealing with decentralized problems is to decompose them into several centralized problems in a reasonable fashion. Our objectives is now to investigate such an approach and elaborate more on some of the important properties of the controllers, under various communication scenarios among the stations.

Consider the system (4.1) again. We would like to design the control algorithms based on two centralized LQG problems. Namely, let each station pretend that it have access to both of the controls, while it only has access to its own information. In other words, the i-th station ($i = 1, 2$) wants to design $u_1^i = u_1^i(z^i)$ and $u_2^i = u_2^i(z^i)$ in order to minimize the following cost:

$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(x^T(t)Qx(t) + {u_1^i}^T(t)R_1u_1^i(t) + {u_2^i}^T(t)R_2u_2^i(t)\right) dt\right] . \qquad (4.5)$$

From the well-known centralized LQG results [11], the optimal controls can be obtained as:

$$\begin{bmatrix} u_1^i(t) \\ u_2^i(t) \end{bmatrix} = \begin{bmatrix} -R_1^{-1}B_1^T\Pi\hat{x}^i(t) \\ -R_2^{-1}B_2^T\Pi\hat{x}^i(t) \end{bmatrix} = \begin{bmatrix} -K_1\hat{x}^i(t) \\ -K_2\hat{x}^i(t) \end{bmatrix} \qquad i = 1, 2 \ , \qquad (4.6)$$

where Π is obtained from the control Riccati equation:

$$-\Pi A - A^T\Pi + \Pi\left(B_1R_1^{-1}B_1^T + B_2R_2^{-1}B_2^T\right)\Pi - Q = 0 \ , \qquad (4.7)$$

and $\hat{x}^i$ is the local state estimate in the i-th station:

$$\dot{\hat{x}}^i(t) = A\hat{x}^i(t) + B_1u_1^i(t) + B_2u_2^i(t) + L_i\left(z^i(t) - H^i\hat{x}^i(t)\right), \qquad i = 1, 2 \ , \quad (4.8)$$

The estimator gain is obtained as:

$$L_i \triangleq P_i(H^i)^T\left(V^i\right)^{-1}, \qquad i = 1, 2 \ , \qquad (4.9)$$

where P_i is the solution to the corresponding filter Riccati equation:

$$AP_i + P_iA^T - P_i(H^i)^T \left(V^i\right)^{-1} H^iP_i + W = 0, \quad i = 1, 2 \; . \tag{4.10}$$

Note that the only difference in the two centralized problems comes from the fact that the stations have access to different information, i.e., from the matrix H^i and the noise intensity matrix V^i.

After solving the two centralized problems, u_1^1 and u_2^2 will be applied to the decentralized system. Obviously, there is no reason for these controllers to be optimal for the decentralized system. Also they are not guaranteed to preserve any level of performance, including even the closed-loop stability. However, in some cases, where the stations are allowed to communicate some pieces of information through low noise channels, we would expect the local stations to obtain very similar controllers, which in turn, are expected to be very close to the decentralized optimal controllers.

4.4 Closed-Loop Stability

Achieving closed-loop stability is one of the most important performance properties that we desire for our controllers. We know that the centralized LQG controllers will always stabilize the system under some *detectability* and *stabilizability* conditions. But in general, there is no reason to guarantee closed-loop stability if we apply the same centralized control algorithms to the decentralized system. In this section, we will investigate the closed-loop stability properties of our controllers in various situations, where the stations communicate different pieces of information. Note that in some cases, based on the available information for each station, we may modify the estimators, and hence deviate a little bit from the original centralized LQG solutions. In such cases, we will actually be looking at general *linear estimate linear feedback* structures.

In order to analyze the dynamics of the closed-loop system, we define the local estimation errors and the difference between the local estimates respectively as:

$$
\begin{aligned}
e_1(t) &\triangleq x(t) - \hat{x}^1(t) && (4.11)\\
e_2(t) &\triangleq x(t) - \hat{x}^2(t) && (4.12)\\
e_{12}(t) &\triangleq \hat{x}^1(t) - \hat{x}^2(t). && (4.13)
\end{aligned}
$$

We can now write:

$$
\begin{aligned}
\dot{x} &= Ax - B_1K_1\hat{x}^1 - B_2K_2\hat{x}^2 + w \\
&= Ax - B_1K_1\left(e_{12} + x - e_2\right) - B_2K_2\left(x - e_2\right) + w \\
&= \left(A - B_1K_1 - B_2K_2\right)x + \left(B_1K_1 + B_2K_2\right)e_2 - B_1K_1e_{12} + w && (4.14)\\
\dot{e}_2 &= Ax - B_1K_1\hat{x}^1 - B_2K_2\hat{x}^2 + w - A\hat{x}^2 + B_1K_1\hat{x}^2 + B_2K_2\hat{x}^2 \\
&\quad -L_2\left(H^2e_2 - v^2\right) \\
&= \left(A - L_2H^2\right)e_2 - B_1K_1e_{12} + w - L_2v^2 && (4.15)\\
\dot{e}_{12} &= \left(A - B_1K_1 - B_2K_2\right)e_{12} + L_1H^1x - L_2H^2x - L_1H^1\hat{x}^1 \\
&\quad +L_2H^2\hat{x}^2 + L_1v^1 - L_2v^2 \\
&= \left(A - B_1K_1 - B_2K_2 - L_1H^1\right)e_{12} + \left(L_1H^1 - L_2H^2\right)e_2 \\
&\quad +L_1v^1 - L_2v^2, && (4.16)
\end{aligned}
$$

Hence, the closed-loop system dynamics can be written as follows:

$$
\begin{bmatrix} \dot{x} \\ \dot{e}_2 \\ \dot{e}_{12} \end{bmatrix} = \tag{4.17}
$$

$$
\begin{bmatrix} A-B_1K_1-B_2K_2 & B_1K_1+B_2K_2 & -B_1K_1 \\ 0 & A-L_2H^2 & -B_1K_1 \\ 0 & L_1H^1-L_2H^2 & A-B_1K_1-B_2K_2-L_1H^1 \end{bmatrix} \begin{bmatrix} x \\ e_2 \\ e_{12} \end{bmatrix} +
$$

$$
\begin{bmatrix} I & 0 & 0 \\ I & 0 & -L_2 \\ 0 & L_1 & -L_2 \end{bmatrix} \begin{bmatrix} w \\ v^1 \\ v^2 \end{bmatrix}.
$$

4.4.1 No Transmission

Assume that each station only has access to its own measurements, i.e., there is no communication between the stations. In this case, the closed-loop dynamics are in the form (4.17), where H^1 and H^2 are the corresponding measurement matrices for the stations, while v^1 and v^2 simply denote the measurement uncertainties.

Let us assume that the stations have the same measurement characteristics. Then it is clear from (4.17) that in order to have a stable closed-loop system, we need to have stable feedback dynamics along with stable local estimators and compensators. We conjecture that these stability properties are sufficient for the closed-loop stability, even if the stations do not have identical measurements. But to achieve such stability properties, we need the global state to be detectable from each local station. This condition, however, is a very strong condition for a decentralized system. In most decentralized systems, the global state can not be detectable from all individual stations. Moreover, even if such a strong condition is satisfied, we still do not have any good justification for our sub-optimal approach in this case. There is really no reason to expect the two centralized controllers to have a good performance if they are applied to the decentralized system.

4.4.2 Control (Estimate) Transmission

In this scenario, the stations communicate only their control values. In other words, each station has access to its own local measurements and the transmitted control of the other station. As we have already mentioned, the communication uncertainties are simply modeled as additive white Gaussian noises. Also all the communications are assumed to be instantaneous, i.e., no communication delay is assumed. Therefore, the information available to the first station is:

$$z_1 = H_1 x + v_1, \qquad u_2(t) + v_{t2}(t), \tag{4.18}$$

while the second station has access to the following information:

$$z_2 = H_2 x + v_2, \qquad u_1(t) + v_{t1}(t), \tag{4.19}$$

where v_{t1} and v_{t2} are the corresponding transmission noises. Each station now incorporates the received control of the other station in its local estimator. Namely, the local estimators are:

$$\dot{\hat{x}}^1(t) = A\hat{x}^1(t) + B_1 u_1(t) + B_2 u_2(t) + B_2 v_{t2}(t) + L_1 \left(z_1(t) - H_1 \hat{x}^1(t)\right) \tag{4.20}$$

$$\dot{\hat{x}}^2(t) = A\hat{x}^2(t) + B_1 u_1(t) + B_1 v_{t1}(t) + B_2 u_2(t) + L_2 \left(z_2(t) - H_2 \hat{x}^2(t)\right), \tag{4.21}$$

where:

$$L_1 \triangleq P_1 H_1^T V_1^{-1} \tag{4.22}$$

$$L_2 \triangleq P_2 H_2^T V_2^{-1}, \tag{4.23}$$

and P_1 and P_2 are still the solutions to the corresponding Riccati equations. Note that P_1 and P_2 are not the local estimation error covariances anymore. The following controls are now applied to the decentralized system:

$$
\begin{aligned}
u_1(t) &= -R_1 B_1^T \Pi \hat{x}^1(t) = -K_1 \hat{x}^1(t) && (4.24)\\
u_2(t) &= -R_2 B_2^T \Pi \hat{x}^2(t) = -K_2 \hat{x}^2(t), && (4.25)
\end{aligned}
$$

where Π is the solution to the corresponding control Riccati equation. It is straightforward to obtain the dynamics of the closed-loop system:

$$
\begin{bmatrix} \dot{x} \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} A - B_1K_1 - B_2K_2 & B_1K_1 & B_2K_2 \\ 0 & A - L_1H_1 & 0 \\ 0 & 0 & A - L_2H_2 \end{bmatrix} \begin{bmatrix} x \\ e_1 \\ e_2 \end{bmatrix}
+ \begin{bmatrix} I & 0 & 0 \\ I & -L_1 & 0 \\ I & 0 & -L_2 \end{bmatrix} \begin{bmatrix} w \\ v^1 \\ v^2 \end{bmatrix} - B_2 \begin{bmatrix} 0 \\ v_{t2} \\ 0 \end{bmatrix} - B_1 \begin{bmatrix} 0 \\ 0 \\ v_{t1} \end{bmatrix}. \quad (4.26)
$$

It is clear that the closed-loop system can be stabilized if the system is stabilizable using both stations and it is detectable from each individual station. As we mentioned earlier, this latter condition can not be satisfied in many decentralized systems. Also even if the control transmission is noiseless, there is still no reason to believe that these centralized controllers are, in any sense, close to the optimal decentralized controllers.

Note that communicating the local estimates is actually equivalent to communicating the control values. This is because we have a cooperative structure. That is, each station can be informed of the control strategy, and specifically the estimator and feedback gains, of the other station *a priori*. Therefore, the stations can simply calculate either the control or the estimate upon receiving the other.

Finally, note that we have incorporated the transmitted controls in the local estimators in a rather straightforward manner. Whether there are better ways to incorporate this new information can be further investigated.

4.4.3 Measurement Transmission

Assume that the stations can communicate all their measurements. In this case, the information available to the stations can be expressed as:

$$z^1(t) \triangleq \begin{bmatrix} z_1^1(t) \\ z_2^1(t) \end{bmatrix} = \begin{bmatrix} H_1 x(t) + v_1(t) \\ H_2 x(t) + v_2(t) + v_{21}(t) \end{bmatrix} \triangleq Hx(t) + v^1(t) \quad (4.27)$$

$$z^2(t) \triangleq \begin{bmatrix} z_1^2(t) \\ z_2^2(t) \end{bmatrix} = \begin{bmatrix} H_1 x(t) + v_1(t) + v_{12}(t) \\ H_2 x(t) + v_2(t) \end{bmatrix} \triangleq Hx(t) + v^2(t), \quad (4.28)$$

where $v_{12}(t)$ and $v_{21}(t)$ are independent transmission noises, which are also assumed to be independent of other underlying uncertainties in the system. Note that in this scenario, both stations have the same information matrix H. Therefore, there cannot be any decentralized fixed modes in this case.

Similar to the previous cases, we solve two separate centralized LQG problems. For the first station we get:

$$\begin{bmatrix} u_1^1(t) \\ u_2^1(t) \end{bmatrix} = \begin{bmatrix} -R_1^{-1} B_1^T \Pi \hat{x}^1(t) \\ -R_2^{-1} B_2^T \Pi \hat{x}^1(t) \end{bmatrix} = \begin{bmatrix} -K_1 \hat{x}^1(t) \\ -K_2 \hat{x}^1(t) \end{bmatrix}, \quad (4.29)$$

where:

$$\dot{\hat{x}}^1(t) = A\hat{x}^1(t) + B_1 u_1^1(t) + B_2 u_2^1(t) + L_1 \left(z^1(t) - H\hat{x}^1(t) \right) \quad (4.30)$$

$$L_1 \triangleq P_1 H^T \left(V^1\right)^{-1}, \qquad AP_1 + P_1 A^T - P_1 H^T \left(V^1\right)^{-1} H P_1 + W = 0 \quad (4.31)$$

$$H \triangleq \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad V^1 \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & V_2 + V_{21} \end{bmatrix}, \quad (4.32)$$

and for the second station:

$$\begin{bmatrix} u_1^2(t) \\ u_2^2(t) \end{bmatrix} = \begin{bmatrix} -R_1^{-1}B_1^T\Pi\hat{x}^2(t) \\ -R_2^{-1}B_2^T\Pi\hat{x}^2(t) \end{bmatrix} = \begin{bmatrix} -K_1\hat{x}^2(t) \\ -K_2\hat{x}^2(t) \end{bmatrix}, \tag{4.33}$$

where:

$$\dot{\hat{x}}^2(t) = A\hat{x}^2(t) + B_1u_1^2(t) + B_2u_2^2(t) + L_2\left(z^2(t) - H\hat{x}^2(t)\right) \tag{4.34}$$

$$L_2 \triangleq P_2H^T\left(V^2\right)^{-1}, \qquad AP_2 + P_2A^T - P_2H^T\left(V^2\right)^{-1}HP_2 + W = 0 \tag{4.35}$$

$$H \triangleq \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad V^2 \triangleq \begin{bmatrix} V_1 + V_{12} & 0 \\ 0 & V_2 \end{bmatrix}. \tag{4.36}$$

In this scenario, we have a very good justification for our sub-optimal approach. Specifically, if the transmissions are noiseless, the two centralized problems will be identical. Therefore, we expect our controllers to be the optimal decentralized controllers, which would preserve all the desired properties, including the closed-loop stability. Furthermore, if the transmissions are noisy but the transmission noise intensities are small, we would still expect the controllers to be close to the optimal stabilizing decentralized controllers. In other words, we would not expect any drastic change in the behavior of the controlled decentralized system upon introducing some small transmission noise.

We shall now look at the closed-loop stability properties. It is easy to obtain the following closed-loop system dynamics, which would be valid for any linear estimate linear feedback structure:

$$\begin{bmatrix} \dot{x} \\ \dot{e}_2 \\ \dot{e}_{12} \end{bmatrix} = \tag{4.37}$$

$$\begin{bmatrix} A-B_1K_1-B_2K_2 & B_1K_1+B_2K_2 & -B_1K_1 \\ 0 & A-L_2H & -B_1K_1 \\ 0 & (L_1-L_2)\,H & A-B_1K_1-B_2K_2-L_1H \end{bmatrix} \begin{bmatrix} x \\ e_2 \\ e_{12} \end{bmatrix}$$

$$+ \begin{bmatrix} I & 0 & 0 \\ I & 0 & -L_2 \\ 0 & L_1 & -L_2 \end{bmatrix} \begin{bmatrix} w \\ v^1 \\ v^2 \end{bmatrix}.$$

We notice that the closed-loop system matrix has an interesting structure. The first diagonal block matrix is simply the matrix associated with the feedback dynamics, which could be stabilized if the system is stabilizable using both control stations. The second diagonal block matrix could also be made stable under a simple detectability condition. That is, if the global state is detectable using both stations. Note that this is a much weaker condition than detectability from each individual station, which would be required if the stations did not communicate their measurements. The third diagonal block matrix, however, is the matrix corresponding to the *compensator* dynamics, which may not be stable.

This is a significant result. Let us assume that the transmission noise intensities are very small. Then the estimator gains would be almost the same and the closed-loop system matrix would be very close to a block upper-triangular matrix. We can see that if the compensator is unstable (which might be the case in many systems, especially those with a non-minimum phase structure), the closed-loop system will become unstable because of the unstable dynamics governing the difference between the estimates of the two local estimators. Actually, even when the transmissions are noiseless, there is still an unstable subsystem corresponding to e_{12}. This does not comply with our initial expectation. Note that when the transmissions are noiseless, there is no forcing input for this unstable subsystem, but any small nonzero e_{12} could propagate to infinity! Such a nonzero difference between the local estimates, which could be generated from any difference in the initial conditions of the local estimators, round off errors, etc., would again induce a non-classical information pattern.

4.4.4 Measurement and Control Transmission

We saw that if the stations communicate only their measurements, our sub-optimal controllers may not be able to stabilize the closed-loop system, even though they will yield the centralized optimal stabilizing controllers, in the limit, when the transmission noise intensities go to zero. In this section, we will see how transmitting the controls along with the measurements will help us stabilize the closed-loop system, using a similar sub-optimal approach.

As in the previous case, assume that the stations transmit their measurements through noisy channels, i.e.:

$$z^1(t) \triangleq \begin{bmatrix} z_1^1(t) \\ z_2^1(t) \end{bmatrix} = \begin{bmatrix} H_1 x(t) + v_1(t) \\ H_2 x(t) + v_2(t) + v_{21}(t) \end{bmatrix} \tag{4.38}$$

$$z^2(t) \triangleq \begin{bmatrix} z_1^2(t) \\ z_2^2(t) \end{bmatrix} = \begin{bmatrix} H_1 x(t) + v_1(t) + v_{12}(t) \\ H_2 x(t) + v_2(t) \end{bmatrix}. \tag{4.39}$$

Also assume that the stations communicate their control values. For a little more generality, assume that the communication uncertainties on the controls are modeled by an additive Gaussian uncertainty along with a scale-factor error. Namely, the first station also has access to $(I + \triangle_2)u^2(t) + v_{t2}(t)$, while the second station receives $(I + \triangle_1)u^1(t) + v_{t1}(t)$. Transmission noises $v_{t1}(t)$ and

$v_{t2}(t)$ are assumed to be independent of each other and also independent of all other uncertainties in the system.

Each station again incorporates the transmitted control of the other station in its local estimator. That is, the estimators are constructed in the following manner:

$$\dot{\hat{x}}^1(t) = A\hat{x}^1(t) + B_1 u^1(t) + B_2 (I + \triangle_2) u^2(t) + B_2 v_{t2}(t) + L_1 \left(z^1(t) - H\hat{x}^1(t)\right) \tag{4.40}$$

$$\dot{\hat{x}}^2(t) = A\hat{x}^2(t) + B_1 (I + \triangle_1) u^1(t) + B_1 v_{t1}(t) + B_2 u^2(t) + L_2 \left(z^2(t) - H\hat{x}^2(t)\right), \tag{4.41}$$

where:

$$L_1 \triangleq P_1 H^T \left(V^1\right)^{-1} \tag{4.42}$$

$$L_2 \triangleq P_2 H^T \left(V^2\right)^{-1}, \tag{4.43}$$

and P_1 and P_2 are obtained from the same Riccati equations as before. Note that P_1 and P_2 are no longer the estimation error covariances. Using the same definitions for the error variables $e_1(t)$ and $e_2(t)$, the closed-loop dynamics may be written as the following:

$$\begin{bmatrix} \dot{x} \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} A - B_1K_1 - B_2K_2 & B_1K_1 & B_2K_2 \\ -B_2K_2\triangle_2 & A - L_1H & B_2K_2\triangle_2 \\ -B_1K_1\triangle_1 & B_1K_1\triangle_1 & A - L_2H \end{bmatrix} \begin{bmatrix} x \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ I & -L_1 & 0 \\ I & 0 & -L_2 \end{bmatrix} \begin{bmatrix} w \\ v^1 \\ v^2 \end{bmatrix} - B_2 \begin{bmatrix} 0 \\ v_{t2} \\ 0 \end{bmatrix} - B_1 \begin{bmatrix} 0 \\ 0 \\ v_{t1} \end{bmatrix}. \tag{4.44}$$

As we can see, when the scale-factor errors $\triangle_1$ and $\triangle_2$ are small, the closed-loop system matrix is nearly block upper-triangular. The first diagonal block matrix can be made stable if the system is stabilizable using both stations. The second and the third diagonal block matrices can also be made stable if (A, H) is detectable.

We conclude that when the stations communicate their controls as well as their measurements, our sub-optimal approach will at least yield a stable closed-loop system, even if there is small scale-factor errors on the control transmissions.

4.4.5 Estimation Residuals Transmission

So far, we have seen that in order to design a set of sub-optimal stabilizing controllers by solving two centralized problems for a two-station decentralized system, and under some reasonable stabilizability and detectability assumptions, the stations need to communicate both their measurements and their control values.

In this section, we investigate the case where the stations communicate their estimation residuals instead of their measurements and controls. In other words, the first station has access to the following information:

$$z_1 = H_1 x + v_1, \qquad \left(z_2 - H_2 \hat{x}^2\right) + v_{t2}, \tag{4.45}$$

while the information available to the second station is:

$$z_2 = H_2 x + v_2, \qquad \left(z_1 - H_1 \hat{x}^1\right) + v_{t1}, \tag{4.46}$$

where v_{t1} and v_{t2} denote the transmission noises. In the previous cases, the linear structure of the estimators and the controllers naturally came out of the two centralized optimal control problems. In this case, however, we will impose a linear structure on our estimation and control such that each station will linearly incorporate the noisy residual of the other station, i.e., for the first station, we have:

$$
\begin{aligned}
u_1^1 &= -K_1 \hat{x}^1 \\
u_2^1 &= -K_2 \hat{x}^1 \\
\dot{\hat{x}}^1(t) &= A\hat{x}^1(t) + B_1 u_1^1(t) + B_2 u_2^1(t) + L_1^1 \left(z_1(t) - H_1 \hat{x}^1(t)\right) + L_2^1 \left(z_2(t) - H_2 \hat{x}^2(t)\right) \\
&\quad + L_2^1 v_{t2}(t),
\end{aligned} \tag{4.47}
$$

while for the second station, we get:

$$
\begin{aligned}
u_1^2 &= -K_1 \hat{x}^2 \\
u_2^2 &= -K_2 \hat{x}^2 \\
\dot{\hat{x}}^2(t) &= A\hat{x}^2(t) + B_1 u_1^2(t) + B_2 u_2^2(t) + L_1^2 \left(z_1(t) - H_1 \hat{x}^1(t)\right) + L_2^2 \left(z_2(t) - H_2 \hat{x}^2(t)\right) \\
&\quad + L_1^2 v_{t1}(t).
\end{aligned} \tag{4.48}
$$

The gains may now be obtained based on some optimality criteria. Note that when the transmission noises v_{t1} and v_{t2} are zero, the local estimators will have exactly the same structure. Therefore, we expect the estimators to have the same gains in the noiseless transmission case, regardless of how the gains are obtained. Also note that each station has linearly incorporated the received estimation residual of the other station. Even though, this simplifies the problem, it is not necessarily the best way of incorporating this new piece of information.

Similarly to the previous cases, it is straightforward to obtain the closed-loop dynamics as the following:

$$\begin{bmatrix} \dot{x} \\ \dot{e}_2 \\ \dot{e}_{12} \end{bmatrix} = \tag{4.49}$$

$$\begin{bmatrix} A-B_1K_1-B_2K_2 & B_1K_1+B_2K_2 & -B_1K_1 \\ 0 & A-L_1^2H_1-L_2^2H_2 & L_1^2H_1-B_1K_1 \\ 0 & (L_2^1-L_2^2)\,H_2 & A-B_1K_1-B_2K_2-(L_1^1-L_1^2)\,H_1 \end{bmatrix} \begin{bmatrix} x \\ e_2 \\ e_{12} \end{bmatrix}$$

$$+\begin{bmatrix} I & 0 & 0 \\ I & -L_1^2 & -L_2^2 \\ 0 & (L_1^1-L_1^2) & (L_2^1-L_2^2) \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} 0 \\ L_1^2 \\ L_1^2 \end{bmatrix} v_{t1} + \begin{bmatrix} 0 \\ 0 \\ L_2^1 \end{bmatrix} v_{t2}.$$

As we can see, when the transmission noise intensities are small, the closed-loop system matrix will be close to a block upper-triangular matrix, which can easily be stabilized when the system is stabilizable using both stations and $\left(A, \begin{bmatrix} H_1^T & H_2^T \end{bmatrix}^T\right)$ is detectable. This shows us that in some sense, the estimation residuals are more valuable than the measurements, and communicating the residuals is enough to stabilize the system by solving two centralized problems.

4.5 Discrete-Time Case

In this section, we will sketch how a similar problem might be formulated for discrete-time systems and how similar results could easily be obtained.

Consider the following decentralized discrete-time system with two stations:

$$x_{k+1} = Ax_k + B_1u_k^1 + B_2u_k^2 + w_k \tag{4.50}$$

$$z_k^1 = H_1x_k + v_k^1 \tag{4.51}$$

$$z_k^2 = H_2x_k + v_k^2. \tag{4.52}$$

For the infinite-horizon problem, the original objective is again to design the controls $u^1 = u^1(z^1)$ and $u^2 = u^2(z^2)$ in order to minimize the following quadratic cost:

$$J = \lim_{N\to\infty} \frac{1}{2N} E\left[\sum_{k=0}^{N} \left(x_k^T Q x_k + {u_k^1}^T R_1 u_k^1 + {u_k^2}^T R_2 u_k^2\right)\right]. \tag{4.53}$$

This is again a decentralized stochastic problem with a non-classical information pattern. Hence, we do not know the optimal strategies yet. However, similar sub-optimal approaches may be considered for the discrete-time case, which would be well justified if the stations are allowed to communicate some pieces of information.

Let us assume that the stations communicate all their measurements and possibly their controls. By solving two local centralized problems, we get the following controllers:

$$u_k^1 = -K_1 A \hat{x}_k^1, \qquad u_k^2 = -K_2 A \hat{x}_k^2, \tag{4.54}$$

where:

$$K \triangleq \left(R + B^T S B\right)^{-1} B^T S,\ K \triangleq \begin{bmatrix} K_1 \\ K_2 \end{bmatrix},\ R \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix},\ B \triangleq \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \tag{4.55}$$

and S is the solution to the discrete-time control Riccati equation:

$$S = Q + A^T S A - A^T S B \left(R + B^T S B\right)^{-1} B^T S A. \tag{4.56}$$

The local estimators are constructed as follows:

$$\hat{x}_{k+1}^1 = A\hat{x}_k^1 + B_1 u_k^1 + B_2 u_k^{21} + L_1\left(z_{k+1}^1 - H\left(A\hat{x}_k^1 + B_1 u_k^1 + B_2 u_k^{21}\right)\right) \tag{4.57}$$

$$\hat{x}_{k+1}^2 = A\hat{x}_k^2 + B_1 u_k^{12} + B_2 u_k^2 + L_2\left(z_{k+1}^2 - H\left(A\hat{x}_k^2 + B_1 u_k^{12} + B_2 u_k^2\right)\right), \tag{4.58}$$

where $H^T \triangleq \begin{bmatrix} H_1^T & H_2^T \end{bmatrix}$ and the measurement vector of each station has been augmented by the transmitted measurement of the other station. The value that is being incorporated for the second control in the local estimator of the first station has been denoted by u^{21}, while u^{12} similarly denotes the value that is being used for the first control in the local estimator of the second station. These values would actually be the transmitted controls if the stations do communicate their controls. Otherwise, they would be obtained directly from the solutions of the two centralized problems.

The estimator gains are obtained as:

$$L_1 \triangleq P_1 H^T \left(H P_1 H^T + V^1\right)^{-1} \tag{4.59}$$

$$L_2 \triangleq P_2 H^T \left(H P_2 H^T + V^2\right)^{-1}, \tag{4.60}$$

where P_1 and P_2 are the solutions to the corresponding discrete-time filter Riccati equations:

$$P_1 = W + A P_1 A^T - A P_1 H^T \left(H P_1 H^T + V^1\right)^{-1} H P_1 A^T \tag{4.61}$$

$$P_2 = W + A P_2 A^T - A P_2 H^T \left(H P_2 H^T + V^2\right)^{-1} H P_2 A^T. \tag{4.62}$$

In order to analyze the stability of the closed-loop system, we again define the error variables:

$$e_k^1 = x_k - \hat{x}_k^1 \tag{4.63}$$

$$e_k^2 = x_k - \hat{x}_k^2. \tag{4.64}$$

Then, we have:

$$x_{k+1} = \left(I - B_1 K_1 - B_2 K_2\right) A x_k + B_1 K_1 A e_k^1 + B_2 K_2 A e_k^2 + w_k \tag{4.65}$$

$$\begin{aligned} e_{k+1}^1 = {} & \left(I - L_1 H\right) A e_k^1 + \left(I - L_1 H\right) B_2 \left(u_k^2 - u_k^{21}\right) + \left(I - L_1 H\right) w_k \\ & - L_1 v_{k+1}^1 \end{aligned} \tag{4.66}$$

$$\begin{aligned} e_{k+1}^2 = {} & \left(I - L_2 H\right) A e_k^2 + \left(I - L_2 H\right) B_2 \left(u_k^1 - u_k^{12}\right) + \left(I - L_2 H\right) w_k \\ & - L_2 v_{k+1}^2 \end{aligned} \tag{4.67}$$

By looking at the above equations, we can see that similar results, as in the continuous-time case, could be obtained for the discrete-time case. Specifically, if the stations do not communicate their control values, these sub-optimal controllers may not be able to stabilize the system, even if the transmissions are noiseless. But when each station transmits its current value of the control to the other stations, the closed-loop system will be stabilized if the system is stabilizable and detectable from all the stations.

4.6 Summary

A two station decentralized LQG problem was formulated, where the local controllers had to be designed based on some local information in order to minimize a single common cost. This problem generally has a non-classical information pattern and the optimal control algorithms are often unknown. One of the first possible sub-optimal approaches is to decompose the problem into separate centralized problems. In this chapter, we investigated such an approach for different communication scenarios between the stations; namely, when the stations communicate their control values, their measurements or both, or their estimation residuals.

We showed that even though our approach is quite reasonable for the case where the stations communicate all their measurements, the designed controllers may fail to stabilize the closed-loop system as soon as the compensator is unstable. Then, we showed how this difficulty can be removed if the stations either communicate both their measurements and their controls or communicate their estimation residuals.

All these results show some of the fundamental differences between the centralized and the decentralized structures. In fact, a basic issue is that when there is some strong coupling among the stations, the controllers need much more information and every station should, at least, be able to gather some information about the control actions of other stations, whereas when the couplings among the stations are weak, much less coordination is required. In this chapter, we showed how communication among the stations can affect the overall performance and even the closed-loop stability in a decentralized system.

In the next part of the thesis, we focus on a specific two-stage decentralized stochastic problem. Namely, we investigate the classic counter-example of Witsenhausen along with its different reformulations. This problem, while being simple, gives us a very insightful perspective of the difficulties involved in designing stochastic optimal control algorithms, under the *decentralization* constraints.

Part III

A Decentralized Stochastic Control Problem

CHAPTER 5

A Two-Stage Decentralized Stochastic System

5.1 Introduction

So far we have emphasized how difficult it can be to find the optimal strategies for different stations in a decentralized system, as soon as it has a general non-classical information pattern. One main difficulty is that the information available to one station may be insufficient to determine the previous actions by other stations, which have affected that information. This will destroy the convexity of the cost function with respect to the strategies, even though it may look convex in the controls.

Non-classical patterns, however, would easily arise in many applications where the information is distributed. This is mainly because of the fact that most of the time, communication among the stations is far from perfect in the sense that the information received by a station is latent and/or noisy.

In 1968, Witsenhausen provided a simple example in [84], where there are only two stations, the underlying uncertainties are Gaussian and the cost is quadratic. The information pattern, however, is non-classical. He established the existence of the optimal design and by proposing a nonlinear set of strategies, showed that no affine strategy could be optimal. This seemingly simple example, which is also called Witsenhausen's counterexample, turned out to be extremely hard. It is still outstanding after more than 30 years. This example, in fact, originated much research on the links between decentralized stochastic control problems and team theory and the effects of different information patterns on decentralized systems. Although it is a very simple example, it shows the main difficulties in dealing with non-classical information patterns.

In this chapter, we will first state the example in its original form. We will also give some explanations for the structure of the problem. Then, in Sections 5.3, 5.4, and 5.5, we provide various reformulations for the problem. We will first reformulate the problem with a classical information pattern, for which we know the optimal solution. This will help us appreciate more the difficulties that arise due to the non-classical nature of the information pattern. Next, we will focus again on the original formulation and discuss some of the results available for this case. As we shall see, the non-classical nature of the information pattern is actually induced by denying a piece of information. We will then give another formulation for the problem, where that piece of information is being transmitted through a noisy channel. In this new formulation, we still have a non-classical information pattern, but we believe it can be of more practical interest. We will obtain an alternative expression for the performance index, which shows that it may not be convex in the strategies, as soon as some uncertainty is involved in the information transmission. We will also discuss the limit cases, where the transmission noise intensity goes to zero or grows to infinity. We will see how our reformulated example covers a very wide range, from a classical LQG problem to the Witsenhausen counter-example. Finally in Section 5.6, we will discuss another formulation in the form of a communications problem, where an information-theoretic approach has been used in order to find the optimal strategies. Concluding remarks are provided in the final section.

In the next chapter, we will present an asymptotic approach to obtain the asymptotically optimal strategies for the case where the transmission noise intensity is small.

5.2 Witsenhausen's Counter-Example

Consider a two-stage stochastic problem with the following state equations:

$$x_1 = x_0 + u_1 \tag{5.1}$$

$$x_2 = x_1 - u_2, \tag{5.2}$$

where x_0 is the random initial state, which is assumed to be Gaussian with zero mean and variance σ_0^2. The information pattern of the system is determined by the following output equations:

$$z_1 = x_0 \tag{5.3}$$

$$z_2 = x_1 + v_2 = x_0 + u_1 + v_2, \tag{5.4}$$

where v_2 is the measurement noise for the second station, which is again assumed to be a zero mean Gaussian random variable with unit variance. It is independent from x_0. The objective is to design the control strategies:

$$u_1 = \gamma_1(z_1) \tag{5.5}$$

$$u_2 = \gamma_2(z_2), \tag{5.6}$$

in order to minimize the following cost function:

$$J = E\left[k^2 u_1^2 + x_2^2\right], \tag{5.7}$$

where $k^2 > 0$ is a given constant. Note that this is a sequential control problem in the sense that the second station acts after the first station. In other words, the order in which the stations apply their control actions does not depend on the uncertainties in the system. We see that the first controller has perfect information but its action is costly. In contrast, the second controller has inexpensive control but noisy information. Since the second station does not know what the first station knew, we do not have perfect recall and hence we have a non-classical pattern. Also, since the information available to the second controller is affected by the action of the first controller, we actually have a dynamic information pattern. At the same time, there is no way for the second controller to find out what the first controller did. Therefore, the pattern is not partially nested. Note that the objective can be written as:

$$\min_{u_1,u_2} J = \min_{\gamma_1,\gamma_2} E\left[k^2\gamma_1^2\left(x_0\right) + \left(x_0 + \gamma_1\left(x_0\right) - \gamma_2\left(x_0 + \gamma_1\left(x_0\right) + v_2\right)\right)^2\right], \tag{5.8}$$

which shows that γ_1 not only enters directly in a quadratic form, but also enters indirectly through γ_2. Thus, in general, we lose convexity in the functional space (γ_1, γ_2), even though J is quadratic in u_1 and u_2.

We can see that denying the access to z_1 for the second controller imposes a non-classical nature on the information pattern. In the next section, we will consider the problem with a classical pattern, where the second station also has access to z_1. Then, we will come back to the original problem and discuss some of the available results.

5.3 Formulation with a Classical Pattern: Noiseless Transmission

Consider the state equations (5.1) and (5.2) again. Assume that z_1 is transmitted to the second station through a noiseless channel, i.e., the two stations now have access to the following information:

$$z_1 = x_0 + v_1 \tag{5.9}$$

$$z_2 = \begin{bmatrix} z_1 \\ x_0 + u_1 + v_2 \end{bmatrix}, \tag{5.10}$$

where $v_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $v_2 \sim \mathcal{N}(0, \sigma_2^2)$ are independent measurement noises for the first and second stations respectively. The objective is again to minimize the same cost as (5.7).

Let us compare this problem with the original example. In this problem, the first station also has noisy information. Since the information available to the second station is still affected by u_1, we still have a dynamic pattern. However, since the second controller knows exactly what the first controller knew, we have perfect recall. Hence, we indeed have a classical information pattern and this is the basic difference with the original example.

Remember that *a priori* coordination is allowed between the controllers, i.e., the stations know each other's strategies beforehand. Since u_2 has access to z_1, the information available to the second controller could equivalently be written as:

$$z_2' = \begin{bmatrix} z_1 \\ x_0 + u_1 + v_2 - \gamma_1(z_1) \end{bmatrix} = \begin{bmatrix} z_1 \\ x_0 + v_2 \end{bmatrix}. \tag{5.11}$$

This converts the information pattern to a static pattern, where the information available to the second station is not affected by the action of the first station anymore. Since the cost is quadratic and the uncertainties are assumed to be Gaussian, linear strategies are optimal. To obtain these optimal strategies, we find the person-by-person optimal strategies, which by convexity are known be team optimal as well (see Section 2.2).

We *guess* the following strategies:

$$\begin{aligned} u_1 &= k_1 z_1 && (5.12) \\ u_2 &= k_2 z_2' = k_{21} z_1 + k_{22}\left(x_0 + v_2\right). && (5.13) \end{aligned}$$

Then:

$$\begin{aligned} u_1^* &= \arg\min_{u_1} E\left[k^2 u_1^2 + \left(u_1 + x_0 - k_{21} z_1 - k_{22}\left(x_0 + v_2\right)\right)^2 \middle| z_1\right] \\ &= \arg\min_{u_1} \left\{k^2 u_1^2 + u_1^2 + 2 u_1 E\left[x_0 - k_{21} z_1 - k_{22}\left(x_0 + v_2\right) \middle| z_1\right]\right. \\ &\qquad \left. + E\left[\left(x_0 - k_{21} z_1 - k_{22}\left(x_0 + v_2\right)\right)^2 \middle| z_1\right]\right\} \\ &= -\frac{1}{k^2+1} E\left[x_0 - k_{21} z_1 - k_{22}\left(x_0 + v_2\right) \middle| z_1\right] \\ &= -\frac{1}{k^2+1}\left[k_{21} + \frac{\sigma_0^2}{\sigma_0^2 + \sigma_1^2}\left(1 + k_{22}\right)\right] z_1, && (5.14) \end{aligned}$$

where we have used:

$$\begin{aligned} E\left[x_0 \left| z_1\right.\right] &= \frac{\sigma_0^2}{\sigma_0^2 + \sigma_1^2} z_1 && (5.15) \\ E\left[v_2 \left| z_1\right.\right] &= 0, && (5.16) \end{aligned}$$

Similarly:

$$\begin{aligned} u_2^* &= \arg\min_{u_2} E\left[k^2 k_1^2 z_1^2 + \left(k_1 z_1 + x_0 - u_2\right)^2 \middle| z_2'\right] \\ &= E\left[k_1 z_1 + x_0 \left| z_2'\right.\right] \\ &= \left(k_1 + l_1\right) z_1 + l_2\left(x_0 + v_2\right), && (5.17) \end{aligned}$$

where:

$$\begin{aligned} l_1 &\triangleq \frac{\sigma_0^2 \sigma_2^2}{\sigma_0^2 \sigma_1^2 + \sigma_0^2 \sigma_2^2 + \sigma_1^2 \sigma_2^2} && (5.18) \\ l_2 &\triangleq \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 \sigma_1^2 + \sigma_0^2 \sigma_2^2 + \sigma_1^2 \sigma_2^2}, && (5.19) \end{aligned}$$

and we have used:

$$\begin{aligned} E\left[z_1 \left| z_2'\right.\right] &= z_1 && (5.20) \\ E\left[x_0 \left| z_2'\right.\right] &= l_1 z_1 + l_2\left(x_0 + v_2\right), && (5.21) \end{aligned}$$

The gains k_1, k_{21} and k_{22} can now be obtained from (5.14) and (5.17). When there is no noise in the first station, i.e., $\sigma_1^2 = 0$, we will have:

$$u_1^* = 0 \tag{5.22}$$

$$u_2^* = z_1, \tag{5.23}$$

and the corresponding minimum cost will be zero.

Therefore, when the second station has access to z_1, the information pattern is classical and we can easily obtain the optimal strategies that are linear for the LQG case.

5.4 Formulation with a Non-classical Pattern: No Transmission

We now go back to the original problem, where there is no transmission, and the second station does not have access to z_1. Also the observation for the first station is noiseless, i.e.,:

$$z_1 = x_0 \tag{5.24}$$

$$z_2 = x_1 + v_2 = x_0 + u_1 + v_2. \tag{5.25}$$

As we said, this is a non-classical information pattern and we indeed have a non-convex optimization problem.

It can be shown [38] that the person-by-person optimal strategies are still linear:

$$u_1^p = k_1 z_1 \tag{5.26}$$

$$u_2^p = k_2 z_2, \tag{5.27}$$

where k_1 and k_2 must satisfy the following set of *nonlinear* equations:

$$k_1 = -\frac{1-k_2}{2k^2+(1-k_2)^2} \tag{5.28}$$

$$k_2 = \frac{(1+k_1)\,\sigma_0^2}{(1+k_1)^2\,\sigma_0^2+1}. \tag{5.29}$$

Remember that person-by-person optimal strategies are no longer team optimal. We now follow an approach proposed by Witsenhausen [84].

Define:

$$f(z_1) \triangleq z_1+\gamma_1(z_1) = x_0+u_1 \tag{5.30}$$

$$g(z_2) \triangleq \gamma_2(z_2) = u_2. \tag{5.31}$$

Then the cost can be expressed as:

$$\begin{aligned} J &= E\left[k^2u_1^2+x_2^2\right] \\ &= E\left[k^2\left(z_1-f(z_1)\right)^2+\left(f(z_1)-g(z_2)\right)^2\right] \\ &\triangleq J(f,g). \end{aligned} \tag{5.32}$$

If we fix the function f, the optimal strategy g will clearly be obtained as the conditional expectation, that is:

$$g^*(z_2) = \arg\min_g J(f,g) = E\left[f(z_1)\,|z_2\right]. \tag{5.33}$$

Substituting back in the cost, we get:

$$\begin{aligned} J^*(f) &\triangleq J(f,g^*) \\ &= k^2E\left[\left(z_1-f(z_1)\right)^2\right]+E\left[\left(f(z_1)-g^*(z_2)\right)^2\right] \\ &= k^2E\left[\left(z_1-f(z_1)\right)^2\right]+E\left[\left(f(z_1)\right)^2\right]-E\left[\left(g^*(z_2)\right)^2\right], \end{aligned} \tag{5.34}$$

where we have used the orthogonality property of the conditional expectation:

$$E\left[\left(f(z_1)-g^*(z_2)\right)g^*(z_2)\right] = 0. \tag{5.35}$$

It is important to note the minus sign in the third term in (5.34). As we shall see, this minus sign can actually destroy the convexity of the cost with respect to the strategies.

The objective is now to express the cost $J^*(f)$ in terms of only one strategy f. In doing so, we use the following lemma, which shows how $g^*(z_2)$ may be expressed in terms of information z_2 and its probability density function.

Lemma 5.1. *The optimal strategy $g^*(z_2)$ can be expressed as:*

$$g^*(z_2) = z_2 + \frac{d}{dz_2} \ln p(z_2), \tag{5.36}$$

where $p(z_2)$ is the probability density function for the information available to the second station.

Proof: We have:

$$\begin{aligned} g^*(z_2) &= \int f(z_1)\, p(z_1 | z_2)\, dz_1 \\ &= \frac{\int f(z_1)\, p(z_1, z_2)\, dz_1}{\int p(z_1, z_2)\, dz_1}, \end{aligned} \tag{5.37}$$

where $p(z_1, z_2)$ is the joint probability density of z_1 and z_2. At the same time, we can write:

$$\begin{aligned} z_2 p(z_1, z_2) + \frac{\partial}{\partial z_2} p(z_1, z_2) &= z_2 p(z_1, z_2) + \frac{\partial}{\partial z_2} p(z_2 | z_1)\, p(z_1) \\ &= z_2 p(z_1, z_2) + \frac{\partial}{\partial z_2} p_{v_2}(z_2 - f(z_1))\, p(z_1) \\ &= z_2 p(z_1, z_2) + \frac{\partial}{\partial z_2} \left(\frac{1}{\sqrt{2\pi}} \exp - \frac{(z_2 - f(z_1))^2}{2} \right) p(z_1) \\ &= f(z_1)\, p(z_1, z_2), \end{aligned} \tag{5.38}$$

Integrating both sides, we get:

$$\int f(z_1)\, p(z_1, z_2)\, dz_1 = z_2 p(z_2) + \frac{d}{dz_2} p(z_2). \tag{5.39}$$

Substituting back in (5.37), we obtain $g^*(z_2)$ as follows:

$$g^*(z_2) = z_2 + \frac{d}{dz_2} \ln p(z_2). \tag{5.40}$$

$\diamondsuit$

As we shall see, when we try to express the cost in terms of only a single strategy f, a *Fisher Information* term comes up in the cost. Fisher information is originally obtained in the Cramer-Rao bound, which is a measure for the minimum error in estimating a parameter based on the value of a random variable. However, by introducing a location parameter, an alternative form of the Fisher information may be defined for a random variable with a given distribution. This alternative form is, in fact, related to the entropy measure (see [16], p.494). We first present the definition for the Fisher information matrix.

Definition 1. *The Fisher information matrix for a random vector Z is defined as:*

$$I_f(Z) \triangleq E\left[\nabla_z^T \ln p(z) \cdot \nabla_z \ln p(z)\right]. \tag{5.41}$$

where $p(z)$ is the probability density function for the random variable Z and ∇_z denotes the gradient vector with respect to z:

$$\nabla_z \triangleq \left[\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n}\right] \tag{5.42}$$

where z_i is the i-th component in the random vector.

We are now ready to express the cost (5.34) only in terms of f.

Theorem 3. *The performance index (5.34) can be written as:*

$$J^*(f) = k^2 E\left[(z_1 - f(z_1))^2\right] + 1 - I_f(Z_2), \tag{5.43}$$

where $I_f(Z_2)$ is the Fisher information *for Z_2.*

(The subscript f indicates the fact that it actually depends on the form of the function f, which affects the information available to the second station z_2 and would thus affect its probability density function.)

Proof: Using (5.36), we first obtain $E\left[(g^*(z_2))^2\right]$. We have:

$$E\left[(g^*(z_2))^2\right]=E\left[z_2^2\right]+2E\left[z_2\frac{d}{dz_2}\ln p(z_2)\right]+E\left[\left(\frac{d}{dz_2}\ln p(z_2)\right)^2\right]. \tag{5.44}$$

At the same time:

$$E\left[z_2^2\right]=E\left[(f(z_1)+v_2)^2\right]=E\left[(f(z_1))^2\right]+1, \tag{5.45}$$

and:

$$\begin{aligned} E\left[z_2\frac{d}{dz_2}\ln p(z_2)\right] &= \int_{-\infty}^{+\infty} z_2\frac{d}{dz_2}\ln(p(z_2))\,p(z_2)\,dz_2 \\ &= z_2 p(z_2)|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty} p(z_2)\,dz_2=-1, \end{aligned} \tag{5.46}$$

where we have assumed z_2 has a finite mean value and therefore the first term becomes zero. The last term in (5.44) is the Fisher information term:

$$I_f(Z_2)=E\left[\left(\frac{d}{dz_2}\ln p(z_2)\right)^2\right]=\int\left(\frac{d}{dz_2}\ln p(z_2)\right)^2 p(z_2)dz_2. \tag{5.47}$$

Using (5.45) and (5.46) and (5.47) in (5.44) and substituting it back in (5.34), we can express the optimum cost only in terms of the function f as:

$$J^*(f)=k^2E\left[(z_1-f(z_1))^2\right]+1-I_f(Z_2). \tag{5.48}$$

$\diamond$

This shows us that in order to minimize the cost, we need to get the lowest possible cost associated with the first station, while we transfer as much information as possible to the second station through the dynamics of the system. The non-convexity of the cost with respect to f can be seen from the above expression as well. It can be shown that the Fisher information term is a convex functional [15]. Therefore, $1-I_f(Z_2)$ is concave and the sum of a convex and a concave functional may not be convex.

Witsenhausen showed that the optimal solution exists, even if x_0 has a general distribution with a finite second moment [84]. He also showed that the optimal *linear* strategies have the following form:

$$f_l^*(z_1) = k_1 z_1 \tag{5.49}$$

$$g_l^*(z_2) = k_2 z_2, \tag{5.50}$$

where:

$$k_2 = \frac{\sigma_0^2 k_1^2}{1+\sigma_0^2 k_1^2}, \tag{5.51}$$

and $t \triangleq \sigma_0 k_1$ is a real root of the equation:

$$(t-\sigma_0)\left(t^2+1\right)^2 + \frac{t}{k^2} = 0. \tag{5.52}$$

He then showed that if one of the strategies is restricted to be affine, the other optimal strategy would also be affine. But then he provided the nonlinear strategies that could achieve a lower cost. Namely, he showed that when $k^2\sigma_0^2 = 1$ and $k \to 0$, the following strategies outperform the best affine strategies:

$$f_w(z_1) = \sigma_0 \mathrm{sgn}(z_1) \tag{5.53}$$

$$g_w(z_2) = \sigma_0 \tanh(\sigma_0 z_2). \tag{5.54}$$

The basic idea behind these nonlinear strategies is the idea of *signaling*. Note that by applying $u_1 = \sigma_0 \mathrm{sgn}(z_1) - z_1$, the state $x_1 = x_0 + u_1$ will have a two point distribution at $\pm\sigma_0$ and for a large σ_0, the measurement $z_2 = x_1 + v_2$ would be a good estimate for x_1 and hence the second controller can almost cancel it. In this way, the first station actually *signals* its own information to the second controller through the dynamics of the system.

An information theoretic approach was later adopted in [9], where the signaling level in $f_w(z_1)$ was optimized from σ_0 to $\sigma_0\sqrt{2/\pi}$, resulting in about 10% improvement in the cost.

Function approximation techniques have also been explored to obtain the optimal solution. In [4] and [5], a neural network, trained by stochastic approximation techniques, was used to approximate $f(z_1)$. It was demonstrated that the optimal $f^*(z_1)$ may not be strictly piecewise, as was suggested by Witsenhausen, but slightly sloped.

Some researchers have tried to attack the problem numerically. A discretized version of the problem was formulated in [40], which was later shown in [59] to be NP-complete and computationally intractable. In [19] a different approach was considered by searching directly in the strategy space using the generalized step functions to approximate $f(z_1)$. Two main assumptions were made about the structure of the optimal strategy f^*. Namely, $f^*(.)$ was assumed to be monotone nondecreasing and symmetric about the origin. Both of these assumptions were also addressed in the original Witsenhausen paper [84]. This approach resulted in about 47% improvement in the performance, compared with [9]. This sample and search technique was further investigated and generalized in [49] and [50], where better approximations for the optimal strategies were obtained and an additional 13% improvment in the cost was achieved.

Finally, an alternative approach to obtain the optimal strategies is the *asymptotic approach*. This approach was used in [13] for the case where σ_0 is small. Using a simple polynomial expansion for $f(z_1)$ up to the fifth order in terms of z_1 (remember that $z_1 = x_0$), it was shown that all the coefficients, except for the one corresponding to the first order term, turn out to be zero in the optimal strategy, i.e., the optimal strategy f will indeed be linear:

$$f(z_1) = \frac{k^2}{1+k^2}\left(1 + \frac{2\sigma_0^2 k^4}{(1+k^2)^3}\right) z_1. \tag{5.55}$$

5.5 Formulation with a Non-classical Pattern: Noisy Transmission

As we saw in the previous sections, when the second station has access to z_1, we have a classical information pattern where, due to the quadratic nature of the cost and the Gaussian uncertainties, the optimal strategies are linear in the information. However, when the access of the second station to z_1 is denied, we have a non-classical information pattern, where finding the optimal strategies is extremely hard.

Let us now assume that the first station transmits z_1 to the second station through a noisy channel. In other words, the second controller has access only to the corrupted z_1. This is probably of more practical interest in different applications where the information is being communicated through noisy channels. Note that as soon as the information gets corrupted, the information pattern becomes non-classical and all the difficulties appear again. By receiving a noisy z_1, the second controller still can not find out what the first controller did.

In this section, we study a reformulation of the Witsenhausen's example where a noisy transmission of z_1 is allowed. We obtain a similar alternative expression for the performance index as in (5.43). We then discuss the two limit cases where the transmission noise intensity goes to zero or grows to infinity. Later in the next chapter, we follow an asymptotic approach to analyze this problem under the assumption of a low noise transmission.

The major difference of this new formulation with the original Witsenhausen counter-example is in the information pattern. Specifically, the information available to the stations is now expressed as:

$$z_1 = x_0 \tag{5.56}$$

$$z_2 = \begin{bmatrix} x_0 + v_t \\ x_0 + u_1 + v_2 \end{bmatrix} \triangleq \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix}, \tag{5.57}$$

where $v_t \sim \mathcal{N}(0, \epsilon^2)$ is the transmission noise, which is assumed to be independent of other uncertainties in the system. Note that the communication uncertainty is modeled again with an additive white Gaussian noise. A simple justification for this model was presented in Section 4.2.

Using the same definitions for f and g, as in (5.30) and (5.31), the cost can still be written in the form of (5.34):

$$J^*(f) \triangleq J(f, g^*) = k^2 E\left[(z_1 - f(z_1))^2\right] + E\left[(f(z_1))^2\right] - E\left[(g^*(z_2))^2\right]. \tag{5.58}$$

Similar to Lemma 5.1, we can prove the following lemma:

Lemma 5.2. *The optimal strategy $g^*(z_2)$ can be expressed as:*

$$g^*(z_2) = \arg\min_g J(f, g) = E[f(z_1)|z_2] = z_{22} + \frac{\partial}{\partial z_{22}} \ln p(z_2), \tag{5.59}$$

where $p(z_2) = p(z_{21}, z_{22})$ is the probability density function for the information available to the second station.

Proof: Similar to (5.38), we have:

$$\begin{aligned} z_{22} p(z_1, z_2) + \frac{\partial}{\partial z_{22}} p(z_1, z_2) &= z_{22} p(z_1, z_2) + \frac{\partial}{\partial z_{22}} p(z_2 | z_1) p(z_1) \\ = z_{22} p(z_1, z_2) &+ \frac{\partial}{\partial z_{22}} p_{(v_t, v_2)} \left(\begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} - \begin{bmatrix} z_1 \\ f(z_1) \end{bmatrix} \right) p(z_1) \\ = z_{22} p(z_1, z_2) &+ \frac{\partial}{\partial z_{22}} \left(\frac{1}{2\pi\epsilon} \exp\left(-\frac{(z_{21} - z_1)^2}{2\epsilon^2} - \frac{(z_{22} - f(z_1))^2}{2} \right) \right) p(z_1) \\ = f(z_1) p(z_1, z_2), & \end{aligned} \tag{5.60}$$

where we have used the specific structure of the information available to the second station, and the fact that $v_t \sim \mathcal{N}(0, \epsilon^2)$ and $v_2 \sim \mathcal{N}(0,1)$ are independent. Then, by substituting $f(z_1)\, p(z_1, z_2)$ from (5.60) in (5.37) and integrating with respect to z_1, the expression in (5.59) is obtained.

$\diamondsuit$

The following theorem shows how the performance index in this case can be expressed only in terms of a single function f.

Theorem 4. *The performance index (5.58) can be written as:*

$$J^*(f) = k^2 E\left[(z_1 - f(z_1))^2\right] + 1 - I_f(Z_2)_{22}, \tag{5.61}$$

where $I_f(Z_2)_{22}$ is, in fact, the $(2,2)$ element of the Fisher information matrix *for the random vector Z_2.*

Proof: Using (5.59), we first obtain $E\left[(g^*(z_2))^2\right]$. We have:

$$E\left[z_{22}^2\right] = E\left[(f(z_1))^2\right] + 1, \tag{5.62}$$

and:

$$E\left[z_{22}\frac{\partial}{\partial z_{22}}\ln p(z_2)\right] = \int\int_{-\infty}^{+\infty} z_{22}\frac{\partial}{\partial z_{22}}\ln\left(p(z_{21}, z_{22})\right) p(z_{21}, z_{22})\, dz_{21}dz_{22}. \tag{5.63}$$

If we integrate by parts with respect to z_{22}, we will get:

$$\int_{-\infty}^{+\infty} z_{22}\frac{\partial}{\partial z_{22}}\ln\left(p(z_{21}, z_{22})\right) p(z_{21}, z_{22})\, dz_{22}$$
$$= z_{22}p(z_{21}, z_{22})\big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} p(z_{21}, z_{22})\, dz_{22} = -p(z_{21}), \tag{5.64}$$

where z_{22} is assumed to have a finite mean value and therefore the first term becomes zero. Hence:

$$E\left[z_{22}\frac{\partial}{\partial z_{22}}\ln p(z_2)\right] = -1. \tag{5.65}$$

Therefore:

$$E\left[(g^*(z_2))^2\right] = -1 + E\left[(f(z_1))^2\right] + I_f(Z_2)_{22} \tag{5.66}$$

where:

$$I_f(Z_2)_{22} = E\left[\left(\frac{\partial}{\partial z_{22}} \ln p(z_2)\right)^2\right] \tag{5.67}$$

Substituting (5.66) back in (5.58), we get (5.61) as an alternative form for representing the performance index.

$\diamondsuit$

The alternative cost expression in (5.61), where a Fisher information term enters with a negative sign, shows the possible non-convexity of the cost in terms of the strategies. This is to emphasize the fact that any uncertainty in the communication between the stations will again result in a non-classical information pattern, where the optimal strategies are very difficult to obtain.

5.5.1 Limit Cases

It is now worthwhile to consider the two limit cases where the transmission noise intensity ϵ becomes negligible or increases to infinity. These limit cases show that our reformulated example covers a very wide range, from a classical LQG problem to the Witsenhausen counter-example.

First consider the case where no uncertainty is involved in communication between the two stations, i.e., $\epsilon = 0$ and hence $z_{21} = z_1$. As we saw in Section 5.4, in this case, we have perfect recall and the information pattern is classical. In fact, we can write:

$$\begin{aligned} p(z_2) &= p(z_{21}, z_{22}) = p(z_{22} | z_{21}) \, p(z_{21}) \\ &= p(z_{22} | z_1) \, p(z_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z_{22} - f(z_1))^2}{2}\right) p(z_1) \,. \end{aligned} \tag{5.68}$$

Then, from (5.59), we will have:

$$g^*(z_2) = f(z_1) = f(z_{21}), \tag{5.69}$$

which could directly be obtained from the original definition for g^*, i.e.,:

$$g^*(z_2) = E\left[f(z_1)|\, z_2\right] = f(z_1), \tag{5.70}$$

because z_1 is exactly known, when z_2 is given. Substituting this back in (5.58), and minimizing with respect to f, we get:

$$g^*(z_2) = f(z_1) = z_1, \tag{5.71}$$

and hence:

$$\begin{aligned} \gamma_1(z_1) &= 0 && (5.72)\\ \gamma_2(z_2) &= z_1, && (5.73) \end{aligned}$$

which were previously obtained in (5.22) and (5.23).

On the other hand, when the transmission noise intensity increases to infinity, z_{21} and z_{22} become independent and we will have:

$$p(z_2) = p(z_{21}, z_{22}) = p(z_{21})\, p(z_{22}). \tag{5.74}$$

The Fisher information term can now be written as:

$$\begin{aligned} I_f(Z_2)_{22} &= \int\int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial z_{22}} \ln p(z_{21}, z_{22})\right)^2 p(z_{21}, z_{22})\, dz_{21} dz_{22} \\ &= \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial z_{22}} \ln p(z_{22})\right)^2 p(z_{22})\, dz_{22} \\ &= I_f(Z_{22}), && (5.75) \end{aligned}$$

which is indeed the Fisher information content of z_{22} only. Hence:

$$J^*(f) = k^2 E\left[(z_1 - f(z_1))^2\right] + 1 - I_f(Z_{22}). \tag{5.76}$$

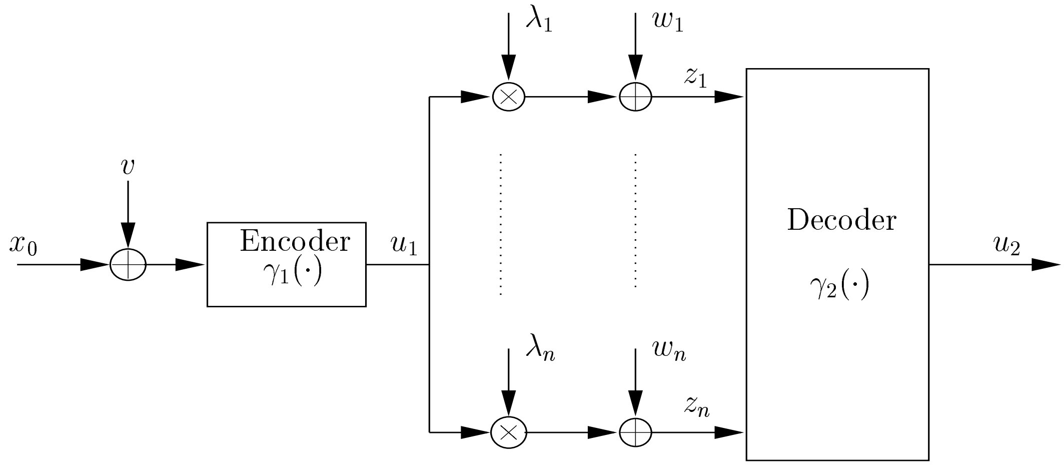


Figure 5.1: Signaling through a multidimensional noisy channel

This is the same result that was presented for the Witsenhausen counter-example (compare with (5.43)). Intuitively, when we have infinite transmission noise intensity, we might as well deny the access to z_1 for the second station, and this is exactly the case in Witsenhausen's counter-example.

5.6 Formulation as a Communications Problem

We have been trying to emphasize the difficulties that arise when the information pattern is a general non-classical pattern. By studying Witsenhausen's example and its various reformulations, we elaborated more on these difficulties and explored some possible approaches in dealing with this class of problems.

In this section, we will discuss a similar problem, which still has a non-classical information pattern and is formulated as a communications problem. This problem was introduced in [8], where an information-theoretic approach was used to show that the optimal strategies are linear.

The problem is to minimize the following cost:

$$J = E\left[k^2 u_1^2 + s u_1 x_0 + (u_2 - x_0)^2\right], \tag{5.77}$$

by designing the optimal strategies:

$$u_1 = \gamma_1(x_0 + v) \tag{5.78}$$

$$u_2 = \gamma_2(z), \tag{5.79}$$

where $x_0 \sim \mathcal{N}(0,1)$ and $v \sim \mathcal{N}(0,\sigma_v^2)$ are independent. Each component of the vector z is represented as:

$$z_i = \lambda_i \gamma_1(x_0 + v) + w_i, \quad \lambda_i \neq 0, \tag{5.80}$$

where w_i's are independent zero mean Gaussian variables with variances $\sigma_{w_i}^2 > 0$.

This problem can actually be interpreted as a communications problem, where a noisy piece of information needs to be transmitted through a multidimensional noisy channel (e.g., a fading channel) with minimum distortion. This is shown in figure (5.1). Note that the hard power constraint on the input is replaced by the term $E[k^2 u_1^2]$ in the cost. A correlation cost $sE[u_1 x_0]$ is also included. The problem is to design the optimal encoder and decoder strategies.

It is clear that the information pattern is non-classical. The decoder obviously does not have access to the information available to the encoder, neither does it have access to the exact encoder output. This is similar to the cases that we discussed before. However, in this problem, the cost function does not contain any product term between the strategies. This is different from the way that the cost was formulated in the previous cases (compare with (5.8)).

Define:

$$m \triangleq E[x_0 \,|x_0 + v] = \frac{1}{1+\sigma_v^2}(x_0 + v) \sim \mathcal{N}(0,\sigma_m). \tag{5.81}$$

Then from the orthogonality principle for the conditional expectation, we can write:

$$E\left[(u_2 - x_0)^2\right] = E\left[(u_2 - m)^2\right] + E\left[(m - x_0)^2\right] \quad (5.82)$$

$$E\left[u_1 x_0\right] = E\left[u_1 m\right]. \quad (5.83)$$

We know that for any fixed strategy γ_1, the decoder strategy γ_2 is the conditional mean:

$$u_2 = \gamma_2(z) = E\left[x_0 \mid z\right]. \quad (5.84)$$

Therefore, the objective of the problem could equivalently be expressed as:

$$\min J = \min_{\gamma_1', \gamma_2} E\left[k^2 {\gamma_1'}^2(m) + sm\gamma_1'(m) + (\gamma_2(z) - m)^2\right] + C, \quad (5.85)$$

where:

$$C \triangleq E\left[(m - x_0)^2\right]. \quad (5.86)$$

The infimum of the above cost under the constraint $E\left[{\gamma_1'}^2\right] = P^2$ can then be obtained, i.e., :

$$J_P \triangleq \inf_{\gamma_1', \gamma_2, E\left[{\gamma_1'}^2\right] = P^2} J \geq k^2 P^2 + \inf sE\left[m\gamma_1'(m)\right] + \inf E\left[(\gamma_2(z) - m)^2\right] + C. \quad (5.87)$$

The second term can be obtained using Cauchy-Schwartz inequality:

$$\inf sE\left[m\gamma_1'(m)\right] = -|s|\sigma_m P. \quad (5.88)$$

But for the third term, information-theoretic concepts have been used. Namely, m, z and $u_2 = \gamma_2(z)$ form a Markov chain and hence the *Data Processing Inequality* [16] applies:

$$I\left(m; u_2\right) \leq I\left(m; z\right), \quad (5.89)$$

where $I(\cdot;\cdot)$ denotes the mutual information, as defined in (3.4). Using the *maximum entropy principle* and (3.4), the following inequalities can be proved [8]:

$$\begin{aligned} I(m;u_2) &\geq \frac{1}{2}\log\frac{\sigma_m^2}{E\left[(\gamma_2(z)-m)^2\right]} && (5.90)\\ I(m;z) &\leq \frac{1}{2}\log\left(1+P^2\lambda\right), && (5.91)\end{aligned}$$

where:

$$\lambda \triangleq \sum_i \frac{\lambda_i^2}{\sigma_{w_i}^2}. \tag{5.92}$$

Thus, we finally get:

$$\begin{aligned} J_P &\geq k^2P^2 - |s|\sigma_m P + \frac{\sigma_m^2}{1+P^2\lambda} + 1 - \sigma_m^2 \\ &\geq k^2P_*^2 - |s|\sigma_m P_* + \frac{\sigma_m^2}{1+P_*^2\lambda} + 1 - \sigma_m^2 \triangleq J_P^*, && (5.93)\end{aligned}$$

where:

$$P_* = \arg\min_P \left(k^2P^2 - |s|\sigma_m P + \frac{\sigma_m^2}{1+P^2\lambda}\right), \tag{5.94}$$

which satisfies the following equation:

$$\left(2k^2P_* - |s|\sigma_m\right)\left(1+P_*^2\lambda\right)^2 = 2P_*\sigma_m^2\lambda. \tag{5.95}$$

At the same time:

$$J^* \geq J_P^*. \tag{5.96}$$

The interesting result that was obtained in [8] was that J_P^* is actually a tight lower bound, which can be achieved by the following linear strategies:

$$\begin{aligned} u_1 &= -\mathrm{sgn}(s)P_*\sigma_m\left(x_0+v\right) && (5.97)\\ u_2 &= E\left[x_0|\, z\right] \\ &= \frac{1}{1+\sigma_v^2}\sum_i \frac{-\lambda_i \mathrm{sgn}(s)P_*\sigma_m}{\sigma_{w_i}^2\left[1/\left(1+\sigma_v^2\right)+\lambda_i P_*^2\sigma_m^2\right]} z_i. && (5.98)\end{aligned}$$

Therefore, the above linear strategies are indeed optimal.

5.7 Summary

In this chapter, we introduced a classical example, known as Witsenhausen's counter-example, which shows some of the fundamental difficulties in decentralized stochastic control problems. We saw that there is a specific piece of information that should be transmitted between the two stations in order to avoid a non-classical information pattern. If this piece of information is denied for the second station, the information pattern becomes non-classical and the problem turns out to be a non-convex functional optimization problem. For the sake of comparison, we explained the same example by allowing a noiseless transmission and hence a classical pattern, where the optimal strategies are well-known.

Then we proposed another formulation for the example, where instead of denying the information, we let it transmit through a Gaussian channel. We asserted that this formulation is closer to real world applications. We followed a similar procedure to convert this problem to an optimization over only one strategy. We then considered the limit cases. We saw that when the transmission noise goes to zero, we will have a classical pattern for which we know the optimal strategies. On the other hand, as the noise power goes to infinity, we will have Witsenhausen's original example.

Finally, we discussed a similar problem, which was formulated as a communications problem. We explored how an information-theoretic approach was used in order to find a tight lower bound for the optimal cost. We saw that the optimal strategies for this specific example are indeed linear, even though the information pattern is non-classical. It was asserted that if the cost does not contain product terms between the decision variables or the controls, the optimal strategies will be linear in the information.

CHAPTER 6

Low Noise Transmission: An Asymptotic Analysis

6.1 Introduction

So far we have shown, through a simple example, how any uncertainty in the transmission of information between the stations in a distributed system can make the optimal control design very complicated and even intractable. Then, by considering the two limit cases, we showed how our example covers a very wide range of scenarios. Namely, we saw that, for the noiseless transmission case, the unique optimal strategies, which are linear in the information, are easily obtained, whereas for the infinite transmission noise intensity, the optimal strategies are still unknown.

We mentioned, in Section 5.4, how an asymptotic approach was used in [13] for the original Witsenhausen problem in order to show that, when the uncertainty on the information available to the first station is small (σ_0 is small), linear strategies are still optimal over a large class of nonlinear strategies. Intuitively, when the uncertainty on the information of the first station is small, the second station will also be able to *guess* what that information was. Therefore, since the problem is cooperative in the sense that the stations are aware of each other's strategies, the second station can almost reconstruct the action of the first station and there is no need for any kind of signaling among the stations through the dynamics of the system. Remember that in Witsenhausen's problem, the non-classical nature of the information pattern is a result of the fact that the information available to the first station is completely inaccessible for the second station. On the other hand, we presented a reformulation of the problem where the first station was allowed to send its information to the second station through a noisy channel, and we showed that as soon as some communication uncertainty is introduced, the information pattern becomes non-classical again and the difficulties in the control design reappear.

Now a very feasible case to investigate for our reformulated example is when the uncertainty on the information transmission is small. In fact, when the transmission noise intensity ϵ is small, one would still expect a similar behavior, as the noiseless transmission case, for the optimal strategies.

In this chapter, we consider this case, that is, we assume a small intensity for v_t. Asymptotic approaches have proved useful for obtaining a better understanding of the structure of the solution for these classes of problems.

One approach would be to start with some expansions for the strategies and try to find the optimal coefficients in the expansions. Remember that by expressing the cost in the form of (5.61), we, in fact, converted the problem into an optimization over only one strategy f. The basic idea is now to find a proper expansion for $f(z_1)$ such that an appropriate corresponding expansion for $J^*(f)$ could be obtained, which is convex with respect to the expansion coefficients. Therefore, the coefficients could then be determined through a simple parameter optimization.

We know that $g^*(z_2) = f^*(z_1) = z_1$ determines the optimal strategies for the noiseless transmission case, i.e., when $\epsilon \to 0$. We are still expecting a symmetric behavior for $f^*(z_1)$ about the origin when the transmission noise is introduced. We are also expecting f^* to be a monotone nondecreasing function, which is indeed proved to be true for the case where there is no transmission or equivalently when $\epsilon \to +\infty$ [84].

One difficulty with this formulation, however, is that we can not say much about the final form of the expansion for the cost $J^*(f)$ by investigating the nature of the expansion for f. This is mainly because of the presence of the complicated Fisher information term.

Therefore, a more reasonable approach is to directly find an appropriate expansion for the cost. This is the basis for our analysis in this chapter. This asymptotic analysis not only gives us insight on how the optimal strategies change as the transmission uncertainty is introduced, but also provides us with a better sense of the complexities in the design procedure.

In Section 6.2, we obtain the first few terms in the expansion of the performance index in terms of the small transmission noise intensity ϵ. Then, in Section 6.3, we use the Hamiltonian approach in order to find a necessary condition for the strategies that minimize the approximated cost. We show that the linear strategies, with slightly different coefficients than the corresponding coefficients for the noiseless transmission case, do indeed satisfy the necessary condition. We provide our concluding remarks in the final section.

6.2 An Expansion for the Cost

Consider again the reformulated example in Section 5.5. The state equations along with the information pattern are repeated here for reference:

$$x_1 = x_0 + u_1 \tag{6.1}$$

$$x_2 = x_1 - u_2, \tag{6.2}$$

$$z_1 = x_0 \tag{6.3}$$

$$z_2 = \begin{bmatrix} x_0 + v_t \\ x_0 + u_1 + v_2 \end{bmatrix} \triangleq \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix}, \tag{6.4}$$

Assume that the first station communicates with the second station through a low noise channel. In other words, the intensity of the white Gaussian transmission noise v_t (i.e., ϵ) is assumed to be small. In this section, we will find an expansion for the cost in terms of ϵ. For this purpose, we first find an expansion for the probability density function of the information available to the second station, i.e., $p(z_2)$. Then, we use (5.59) in order to find the corresponding expansion for $g^*(z_2)$. By substituting back in (5.58), we will obtain the expanded cost only in terms of f.

The probability density function for z_2 can be written as follows:

$$\begin{aligned}
p_\epsilon(z_2) &\triangleq p(z_2) = \int_{-\infty}^{+\infty} p(z_{22}, z_{21}, z_1)\, dz_1 && (6.5)\\
&= \int_{-\infty}^{+\infty} p(z_{22}|z_{21}, z_1)\, p(z_{21}|z_1)\, p(z_1)\, dz_1 && (6.6)\\
&= \int_{-\infty}^{+\infty} p(z_{22}|z_1)\, p(z_{21}|z_1)\, p(z_1)\, dz_1 && (6.7)\\
&= \int_{-\infty}^{+\infty} p(z_{22}|z_1)\, p_{v_t}(z_{21}-z_1)\, p(z_1)\, dz_1 && (6.8)\\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z_{22}-f(z_1))^2}{2}\right) \frac{1}{\sqrt{2\pi}\epsilon} \exp\left(-\frac{(z_{21}-z_1)^2}{2\epsilon^2}\right)\\
&\qquad \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{z_1^2}{2\sigma_0^2}\right) dz_1, && (6.9)
\end{aligned}$$

where for (6.7) we have used the facts that the σ-fields generated by $\{z_{21}, z_1\}$ and $\{z_1, v_t\}$ are the same and z_1, v_t and v_2 are mutually independent.

For small ϵ, we now approximate $\ln p_\epsilon(z_2)$ by considering only the first three terms of its expansion around $\epsilon = 0$. Namely:

$$\ln p_\epsilon(z_2) \simeq \ln p_0(z_2) + \frac{\partial}{\partial\epsilon} \ln p_\epsilon(z_2)\Big|_{\epsilon=0} \epsilon + \frac{\partial^2}{\partial\epsilon^2} \ln p_\epsilon(z_2)\Big|_{\epsilon=0} \epsilon^2. \qquad (6.10)$$

By making the following change of variables:

$$\epsilon y \triangleq z_1 - z_{21} \Rightarrow \epsilon dy = dz_1, \qquad (6.11)$$

we can write $p_\epsilon(z_2)$ in the following form:

$$\begin{aligned}
p_\epsilon(z_2) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(z_{22}-\bar{f}_\epsilon(y)\right)^2}{2}\right) \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(z_{21}+\epsilon y)^2}{2\sigma_0^2}\right)\\
&\qquad \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy, && (6.12)
\end{aligned}$$

where:

$$\bar{f}_\epsilon(y) \triangleq f(\epsilon y + z_{21}). \qquad (6.13)$$

It is now clear that:

$$p_0(z_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z_{22} - f(z_{21}))^2}{2}\right) \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{z_{21}^2}{2\sigma_0^2}\right), \tag{6.14}$$

and hence:

$$\ln p_0(z_2) = -\frac{(z_{22} - f(z_{21}))^2}{2} - \frac{z_{21}^2}{2\sigma_0^2} + \ln\left(\frac{1}{2\pi\sigma_0}\right). \tag{6.15}$$

For the first order term, we have:

$$\left.\frac{\partial}{\partial\epsilon} \ln p_\epsilon(z_2)\right|_{\epsilon=0} = \frac{1}{p_0(z_2)} \left.\frac{\partial}{\partial\epsilon} p_\epsilon(z_2)\right|_{\epsilon=0}. \tag{6.16}$$

At the same time:

$$\begin{aligned}
\left.\frac{\partial}{\partial\epsilon} p_\epsilon(z_2)\right|_{\epsilon=0} &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial\epsilon} \left.\left\{\frac{1}{\sqrt{2\pi}} e^{-\frac{(z_{22}-\bar{f}_\epsilon(y))^2}{2}} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(z_{21}+\epsilon y)^2}{2\sigma_0^2}}\right\}\right|_{\epsilon=0} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} (z_{22} - f(z_{21}))\, y f'(z_{21})\, e^{-\frac{(z_{22}-f(z_{21}))^2}{2}} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{z_{21}^2}{2\sigma_0^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&+ \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z_{22}-f(z_{21}))^2}{2}} \frac{1}{\sqrt{2\pi}\sigma_0} \left(-\frac{z_{21}}{\sigma_0^2}\right) y e^{-\frac{z_{21}^2}{2\sigma_0^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= 0.
\end{aligned} \tag{6.17}$$

Therefore:

$$\left.\frac{\partial}{\partial\epsilon} \ln p_\epsilon(z_2)\right|_{\epsilon=0} = 0. \tag{6.18}$$

We could somehow expect this result. This is because we would expect the behavior of $p_\epsilon(z_2)$ only to depend on the variance of the Gaussian transmission noise, i.e., ϵ^2. Using (6.18), we can now obtain the second order term as:

$$\left.\frac{\partial^2}{\partial\epsilon^2} \ln p_\epsilon(z_2)\right|_{\epsilon=0} = \frac{1}{p_0(z_2)} \left.\frac{\partial^2}{\partial\epsilon^2} p_\epsilon(z_2)\right|_{\epsilon=0}. \tag{6.19}$$

After some tedious but straightforward manipulations, we will get:

$$\begin{aligned}
\left.\frac{\partial^2}{\partial\epsilon^2} \ln p_\epsilon(z_2)\right|_{\epsilon=0} &= -f'^2(z_{21}) + f''(z_{21})(z_{22} - f(z_{21})) + f'^2(z_{21})(z_{22} - f(z_{21}))^2 \\
&\quad +2f'(z_{21})(z_{22} - f(z_{21}))\left(-\frac{z_{21}}{\sigma_0^2}\right) - \frac{1}{2\sigma_0^2} + \frac{z_{21}^2}{\sigma_0^4}.
\end{aligned} \tag{6.20}$$

We can now obtain a second order approximation for $\ln p_\epsilon(z_2)$ by substituting the corresponding terms from (6.15), (6.18) and (6.20) back into the expansion (6.10). In the next step, we substitute the expansion for $\ln p_\epsilon(z_2)$ in (5.59) in order to find the corresponding expansion for $g^*(z_2)$. Remember that $g^*(z_2)$ is the optimal strategy for the second station assuming that the first station has a fixed strategy $\gamma_1(z_1) = f(z_1) - z_1$. We have:

$$\begin{aligned}
g^*(z_2) &= z_{22} + \frac{\partial}{\partial z_{22}} \ln p(z_2) \\
&\simeq z_{22} + \frac{\partial}{\partial z_{22}} \ln p_0(z_2) + \epsilon^2 \frac{\partial}{\partial z_{22}} \left(\frac{\partial^2}{\partial \epsilon^2} \ln p_\epsilon(z_2) \Big|_{\epsilon=0} \right) \\
&= z_{22} - (z_{22} - f(z_{21})) \\
&\quad + \epsilon^2 \left[f''(z_{21}) + 2f'^2(z_{21})(z_{22} - f(z_{21})) + 2f'(z_{21}) \left(-\frac{z_{21}}{\sigma_0^2} \right) \right]. \qquad (6.21)
\end{aligned}$$

Our goal is to get an expansion for the cost, which as we know from (5.58), can be written as:

$$J^*(f) = k^2 E\left[(z_1 - f(z_1))^2\right] + E\left[(f(z_1))^2\right] - E\left[(g^*(z_2))^2\right]. \qquad (6.22)$$

Using the expansion for $g^*(z_2)$ from (6.21), we will have:

$$\begin{aligned}
E\left[(g^*(z_2))^2\right] &\simeq E\left[(f(z_{21}))^2\right] \\
+2\epsilon^2 E&\left[f(z_{21}) \left(f''(z_{21}) + 2f'^2(z_{21})(z_{22} - f(z_{21})) + 2f'(z_{21}) \left(-\frac{z_{21}}{\sigma_0^2} \right) \right) \right], \qquad (6.23)
\end{aligned}$$

where we have neglected the fourth order term in ϵ. Substituting this expansion back in (6.22), we will obtain the following expansion for the cost:

$$\begin{aligned}
J^*(f) &= k^2 E\left[(z_1 - f(z_1))^2\right] + E\left[(f(z_1))^2\right] - E\left[(f(z_{21}))^2\right] \\
-2\epsilon^2 E&\left[f(z_{21}) \left(f''(z_{21}) + 2f'^2(z_{21})(z_{22} - f(z_{21})) + 2f'(z_{21}) \left(-\frac{z_{21}}{\sigma_0^2} \right) \right) \right]. \qquad (6.24)
\end{aligned}$$

Note that when the transmission is noiseless, i.e., $\epsilon = 0$ and therefore $z_{21} = z_1$, we have:

$$J^*(f) = k^2 E\left[(z_1 - f(z_1))^2\right], \qquad (6.25)$$

and $f(z_1) = z_1$ is the obvious unique optimal solution. The above expansion, however, is not exactly in our desired form yet. This is because the third term on the right hand side, which is an average over z_{21}, still depends on ϵ. We shall now rewrite the expansion in (6.24) by explicitly expressing the expectations based on the corresponding probability densities:

$$
\begin{aligned}
J^*(f) = &\int_{-\infty}^{+\infty} \left[k^2 (t - f(t))^2 + f^2(t)\right] \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt \\
- &\int_{-\infty}^{+\infty} \left[f^2(t) + 2\epsilon^2 \left(f(t)f''(t) - 2f(t)f'(t)\frac{t}{\sigma_0^2}\right)\right] \frac{1}{\sqrt{2\pi\left(\sigma_0^2+\epsilon^2\right)}} e^{-\frac{t^2}{2\left(\sigma_0^2+\epsilon^2\right)}} dt \\
- &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 4\epsilon^2 f(t)f'^2(t)\left(\tau - f(t)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tau-f(t))^2}{2}} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt d\tau,
\end{aligned} \tag{6.26}
$$

where we have substituted $p(z_2) = p(z_{22}, z_{21}) \simeq p_0(z_2)$ in the third term, since the higher order terms would be multiplied by ϵ^2 and would then be neglected. Now, the third term turns out to be zero, because:

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 4\epsilon^2 f(t)f'^2(t)\left(\tau - f(t)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tau-f(t))^2}{2}} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt = \\
&\int_{-\infty}^{+\infty} 4\epsilon^2 f(t)f'^2(t) \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} \left(\int_{-\infty}^{+\infty} \left(\tau - f(t)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tau-f(t))^2}{2}} d\tau\right) dt \\
&= 0.
\end{aligned} \tag{6.27}
$$

At the same time, we can expand the probability density of z_{21} up to the second order in ϵ. It is actually straightforward to obtain:

$$
\frac{1}{\sqrt{2\pi\left(\sigma_0^2+\epsilon^2\right)}} e^{-\frac{t^2}{2\left(\sigma_0^2+\epsilon^2\right)}} \simeq \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} + \epsilon^2 \frac{1}{\sqrt{2\pi}\sigma_0^5}\left(t^2 - \sigma_0^2\right) e^{-\frac{t^2}{2\sigma_0^2}} \tag{6.28}
$$

Substituting (6.27) and the above expansion back in (6.26) and neglecting the higher order terms in ϵ, we can finally get the following expansion for the cost:

$$
\begin{aligned}
J^*(f) \;\; = \;\; &\int_{-\infty}^{+\infty} \left[k^2 (t - f(t))^2\right] \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt \\
+\epsilon^2 &\int_{-\infty}^{+\infty} \left[4f(t)f'(t)\frac{t}{\sigma_0^2} - 2f(t)f''(t) + f^2(t)\frac{\sigma_0^2 - t^2}{\sigma_0^4}\right] \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt \\
\triangleq &\; J_0^* + \epsilon^2 J_1^*.
\end{aligned} \tag{6.29}
$$

The objective is now to obtain the function f which minimizes the above approximated cost. In the next section, we use a variational approach in order to find a necessary condition for such a function and show how the linear strategies still satisfy this necessary condition.

6.3 Minimizing the Approximated Cost

So far, we have obtained an expansion for the cost assuming that the transmission noise intensity is small. We have indeed approximated the cost by including only up to the second order term in ϵ. We should now try to minimize this approximated cost in order to find the asymptotically optimal f^*. Obviously, this strategy would be optimal only for a small transmission noise intensity. However, it would still be very helpful for the analysis of the behavior of the optimal strategies when we deviate a little bit from the classical information pattern by introducing a small communication uncertainty.

We now use the Hamiltonian approach in order to find the necessary conditions for the function $f(t)$, which minimizes our approximated cost. For simplicity, let us denote:

$$x_1(t) \triangleq f(t) \tag{6.30}$$

$$x_2(t) \triangleq \dot{x}_1(t) = f'(t) \tag{6.31}$$

$$u(t) \triangleq \dot{x}_2(t) = \ddot{x}_1(t) = f''(t) \tag{6.32}$$

$$p(t) \triangleq \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} \tag{6.33}$$

The Hamiltonian is then defined as [11]:

$$\begin{aligned} \mathcal{H} \ = \ & k^2 \left(t - x_1(t)\right)^2 p(t) + \epsilon^2 \left(4x_1(t)x_2(t)\frac{t}{\sigma_0^2} - 2x_1(t)u(t) + x_1^2(t)\frac{\sigma_0^2 - t^2}{\sigma_0^4}\right) p(t) \\ & + \lambda_1(t)x_2(t) + \lambda_2(t)u(t), \end{aligned} \tag{6.34}$$

where λ_1 and λ_2 are the Lagrange multipliers that should satisfy:

$$\begin{aligned}
\dot{\lambda}_1(t) &= -\mathcal{H}_{x_1} \\
&= \left(2k^2\left(t - x_1(t)\right) - 4\epsilon^2 x_2(t)\frac{t}{\sigma_0^2} - 2\epsilon^2 x_1(t)\frac{\sigma_0^2 - t^2}{\sigma_0^4} + 2\epsilon^2 u(t)\right)p(t) \qquad (6.35) \\
\dot{\lambda}_2(t) &= -\mathcal{H}_{x_2} \\
&= -4\epsilon^2 x_1(t)\frac{t}{\sigma_0^2}p(t) - \lambda_1(t). \qquad (6.36)
\end{aligned}$$

But as we can see, the Hamiltonian is linear in $u(t)$, and we actually have a *singular* optimization problem. The singular surface will be characterized by setting $\mathcal{H}_u$ and its derivatives with respect to t equal to zero, that is:

$$\mathcal{H}_u = -2\epsilon^2 x_1(t)p(t) + \lambda_2(t) = 0, \tag{6.37}$$

and:

$$\frac{d}{dt}\mathcal{H}_u = -2\epsilon^2 \dot{x}_1(t)p(t) - 2\epsilon^2 x_1(t)\dot{p}(t) + \dot{\lambda}_2(t) = 0. \tag{6.38}$$

Substituting $\dot{p}(t) = -\frac{t}{\sigma_0^2}p(t)$ and also $\dot{\lambda}_2$ from (6.36), we will get:

$$\frac{d}{dt}\mathcal{H}_u = -2\epsilon^2 x_2(t)p(t) - 2\epsilon^2 x_1(t)\frac{t}{\sigma_0^2}p(t) - \lambda_1(t) = 0. \tag{6.39}$$

Differentiating again and substituting $\dot{\lambda}_1$ from (6.35), we will have:

$$\frac{d^2}{dt^2}\mathcal{H}_u = -4\epsilon^2 u(t)p(t) + 4\epsilon^2\frac{t}{\sigma_0^2}x_2(t)p(t) - 2k^2\left(t - x_1(t)\right)p(t) = 0. \tag{6.40}$$

Therefore, the corresponding $u(t)$ on the singular surface is:

$$u(t) = x_2(t)\frac{t}{\sigma_0^2} - \frac{k^2}{2\epsilon^2}\left(t - x_1(t)\right). \tag{6.41}$$

Note that the first order *generalized Legendre-Clebsch* condition, which is a necessary condition for $u(t)$ to be minimizing on the singular surface, is also satisfied, namely:

$$\frac{\partial}{\partial u}\left(\frac{d^2}{dt^2}\mathcal{H}_u\right) \leq 0, \tag{6.42}$$

Therefore, the corresponding $x_1(t)$ and $x_2(t)$, which minimize our approximated cost, should necessarily satisfy the following differential equations:

$$\dot{x}_1(t) = x_2(t) \tag{6.43}$$

$$\dot{x}_2(t) = x_2(t)\frac{t}{\sigma_0^2} - \frac{k^2}{2\epsilon^2}\left(t - x_1(t)\right) \tag{6.44}$$

Since ϵ is assumed to be small, we may assume the following form in order to obtain the solutions for the above differential equations:

$$x_1(t) = a_0(t) + \epsilon^2 a_2(t) + \epsilon^4 a_4(t) + \ldots \tag{6.45}$$

$$x_2(t) = b_0(t) + \epsilon^2 b_2(t) + \epsilon^4 b_4(t) + \ldots \tag{6.46}$$

Interestingly enough, by substituting the above x_1 and x_2 back into the differential equations and comparing the coefficients of the terms with the same order in ϵ, we will get:

$$x_1(t) = \left[1 - \frac{2\epsilon^2}{k^2\sigma_0^2} + \left(\frac{2\epsilon^2}{k^2\sigma_0^2}\right)^2 - \left(\frac{2\epsilon^2}{k^2\sigma_0^2}\right)^3 + \ldots\right] t = \frac{t}{\left(1 + \frac{2\epsilon^2}{k^2\sigma_0^2}\right)}. \tag{6.47}$$

Back to our original notation, we indeed have:

$$f\left(z_1\right) = \frac{z_1}{\left(1 + \frac{2\epsilon^2}{k^2\sigma_0^2}\right)}. \tag{6.48}$$

As we can see, the solution is still linear with a coefficient which is slightly different than the corresponding coefficient for the noiseless transmission case. Remember that $f\left(z_1\right) = z_1$ is the optimal solution when there is no transmission noise, and note that for $\epsilon = 0$ in (6.48), we get exactly the same solution, as one would expect. Given the above function $f\left(z_1\right)$, the corresponding $g^*\left(z_2\right)$ can easily be obtained using (5.59). Note that it will also be linear because of the Gaussian assumption for the underlying uncertainties.

We could somehow expect the optimal strategies to be linear from the beginning. As we mentioned in Section 6.1, linear strategies were shown to be asymptotically optimal for the Witsenhausen example when the uncertainty on the information available to the first station is small [13]. In this chapter, however, we have considered a reformulation of Witsenhausen's problem where the first station sends its information to the second station through a low noise channel. These two scenario are somewhat similar. Namely, in both scenarios, the second station can determine the information available to the first station fairly accurately. Specifically, in the first scenario, the second station almost knows z_1 because of its small uncertainty, while in the second scenario, it can determine z_1 from the information that is transmitted through a low noise channel.

We would also expect the optimal strategies to approach the corresponding strategies for the noiseless transmission case as the value of z_1 and, in some sense, the *signal to noise ratio* increases. This does not seem to happen in the solution (6.48). One may justify this by looking at the exponential function in the cost (6.29). This function drives the integrand of the cost to zero exponentially fast for large values of z_1. Therefore, the structure of the cost really does not force the optimal solution to approach $f(z_1) = z_1$ as z_1 increases.

We shall now obtain the corresponding value of the cost. Substituting $f(t)$ from (6.48) back into the cost (6.29), we get:

$$
\begin{aligned}
J^*(f) &= \int_{-\infty}^{+\infty} \left[k^2 \left(t - \frac{t}{1+\frac{2\epsilon^2}{k^2\sigma_0^2}} \right)^2 \right] \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt \\
&+\epsilon^2 \int_{-\infty}^{+\infty} \left[4 \frac{t}{\left(1+\frac{2\epsilon^2}{k^2\sigma_0^2}\right)^2} \frac{t}{\sigma_0^2} + \frac{t^2}{\left(1+\frac{2\epsilon^2}{k^2\sigma_0^2}\right)^2} \frac{\sigma_0^2 - t^2}{\sigma_0^4} \right] \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt \\
&= \frac{1}{\left(1+\frac{2\epsilon^2}{k^2\sigma_0^2}\right)^2} \left(2\epsilon^2 + \frac{4\epsilon^4}{k^2\sigma_0^2} \right) \\
&\simeq 2\epsilon^2 - \frac{4\epsilon^4}{k^2\sigma_0^2} \; , \qquad (6.49)
\end{aligned}
$$

where we have used:

$$\int_{-\infty}^{+\infty} t^2 \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt = \sigma_0^2 \quad (6.50)$$

$$\int_{-\infty}^{+\infty} t^4 \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{t^2}{2\sigma_0^2}} dt = 3\sigma_0^4. \quad (6.51)$$

The optimal cost for the noiseless transmission case is zero. But if we use $f(z_1) = z_1$ when the transmission is noisy, we get the following cost:

$$J^*(f) = 2\epsilon^2. \quad (6.52)$$

In other words, if we fix the strategies to be the optimal strategies for the noiseless transmission case, while we introduce a small transmission noise, the increase in the cost will be proportional to the transmission noise intensity. However, if we use (6.48), we can indeed improve the cost by the fourth order in ϵ.

One should note from (6.48) and (6.49) that as the value of $k^2\sigma_0^2$ increases, the asymptotically optimal solution approaches $f(z_1) = z_1$ and the change in the cost becomes smaller. In other words, increasing $k^2\sigma_0^2$ has a similar effect as decreasing the communication uncertainty. To explain this, we note from the performance index that increasing k^2 implies a more expensive control action for the first station, which, in turn, results in smaller u_1. This then implies that the information available to the second station is less affected by the action of the first station. At the same time, increasing σ_0^2 implies a higher level of uncertainty on x_0, which, incidentally, is the piece of information that is being transmitted between the stations.

This again brings up the concept of *information value* and how it could be different for control and communication purposes. In fact, we know from information theory that a higher level of uncertainty for a piece of information implies a higher level of *entropy* and therefore a more valuable piece of information for transmission. On the other hand, however, a more uncertain piece of information would probably be less valuable for control purposes and would have smaller effect on the control strategies. In other words, a control designer would probably be willing to spend less on installing transmitters on the stations for communicating more uncertain pieces of information. We elaborated more on this issue in Chapter 3.

6.4 Summary

We considered a reformulation of the Witsenhausen counter-example, where the first station is allowed to communicate its information with the second station. We assumed the communication uncertainty is small, and followed an asymptotic approach where we approximated the cost based on its expansion in terms of the small transmission noise intensity. We showed how minimizing the approximated cost can be seen as a singular optimization problem. We then used a variational approach in order to find a necessary condition for the asymptotically optimal strategies, and showed that some reasonable linear strategies do actually satisfy the necessary condition. We also provided some intuitive explanations for the behavior of those linear strategies and obtained the corresponding cost.

All the derivations and the results in this chapter clearly show some of the difficulties involved in dealing with decentralized systems, as soon as we deviate a little bit from a classical, or at least a partially nested, information pattern. On the other hand, even though we have modeled the communication uncertainty in the simplest possible way, we have tried to emphasize the role of communication uncertainties in generating such information patterns that are very difficult to handle.

Finally, it should be mentioned that even though the optimization problem is generally difficult for this class of systems, in some applications, one might be able to exploit the specific structure of the system in order to obtain some reasonably good *sup-optimal* strategies, which could yield an acceptable performance.

Part IV

Power Control: A Stocahstic Decentralized Problem in Wireless Communication

CHAPTER 7

Power Control for Cellular Wireless Systems

7.1 Introduction

The rapid growth of wireless communication systems along with the ever increasing need for capacity with limited amount of resources have made the optimal resource allocation one of the biggest challenges for system designers. Power control for wireless systems is one such challenge, which has attracted a lot of attention in recent years. The main idea is to control the transmit power level of a user in a wireless system in order to maintain an acceptable level of quality of service, while eliminating unnecessary interference to other users in the network. Different objectives and approaches have been perceived for power control and different algorithms have been naturally obtained.

The major objective in Direct Sequence Code Division Multiple Access systems is to mitigate the multiple access interference and therefore the *near-far* effect, whereas in Time/Frequency Division Multiple Access systems the objective is mostly to control the co-channel interference. Power control will also minimize the power consumption for the users and hence prolong their battery life.

We focus on power control algorithms that are based on Signal to Interference plus Noise Ratio (SIR). Note that the Bit Error Rate (BER) or the Frame Error Rate (FER) is usually considered as a measure for the Quality of Service (QoS) in a network. On the other hand, we may write:

$$P(e) = \int P(e|r)P(r)dr \tag{7.1}$$

where $P(e)$ is the bit error probability and r is the received SIR per bit. Now $P(e|r)$, which is the bit error probability for a given SIR, would depend on the specific signaling scheme, i.e., modulation, coding, etc. In general, however, higher SIR would yield better bit error performance and it is therefore common to abstract the system architecture and consider SIR as the measure for quality of service in order to formulate the power control objective[1].

We briefly mentioned, in Section 2.6.2, how power control problem can be considered as a decentralized stochastic problem. In fact, if we consider a cellular network on a single channel, every co-channel user acts as a local station. Such local stations are assumed to have access to noisy measurements of their own SIRs, while they are all coupled through the interference that they are causing for each other.

In the next section, we introduce *decision-feedback* and *information-feedback* algorithms as the two main categories of power control algorithms. In Section 7.3 we review the two main approaches for information-feedback power control design, i.e., *SIR balancing* and *SIR threshold* approaches. It has been recognized that SIR balancing could serve as a performance bound for the SIR threshold approach if all the users in the network were to have the same desired SIR. In other words, the optimal balanced SIR is the best SIR that *all* users in the network can simultaneously achieve if no receiver noise is considered. We will elaborate more on the relations between these two approaches and, in fact, formally show how the two approaches can be unified.

[1]In practice, an outer control loop may be implemented where, based on the desired quality of service and the measured BER or FER, the target SIR is adaptively adjusted for an inner power control loop

Then, in Section 7.4, we focus on the feasibility condition in the SIR threshold approach. The feasibility condition is expressed in terms of an upper bound on the spectral radius of a matrix formed from the channel gains and the desired SIR thresholds. In fact, it is already shown how this spectral radius can be seen as a congestion measure for a network [32] and how it can be used for call admission control purposes [33][80]. We will provide a sufficient condition for feasibility in terms of upper bounds on the individual desired SIR thresholds, which are calculated based on the channel gains. Even though this condition is only sufficient and maybe conservative, it is easier to verify. It also confirms the intuitive result that higher SIR thresholds can be supported for the networks with more diagonally dominant channel gain matrices, which in turn, implies weaker couplings among the users in the network.

Finally, it has been noticed that the widely proposed distributed power control algorithm is simply an integrator algorithm in the logarithmic scale. This has initiated a new approach for power control design where a decentralized regulator formulation has been proposed and concepts and design methodologies from control theory have been used for the analysis of current algorithms [68] and design of new algorithms [21][31]. We present this formulation in Section 7.5.

This approach could be specifically helpful if models for fading, i.e., channel gain variations, are to be incorporated in the design. However, stability and convergence of these algorithms cannot be verified through simple techniques such as the one presented in [87]. Therefore more complicated stability analysis should be performed to ensure *global stability* of the network under these power control algorithms. In Section 7.6, we investigate how one could analyze the global stability of a network on a single channel.

A robust control framework was presented in [31], where a sufficient condition for global stability was established using a linearized interference function. We use a similar framework to obtain another sufficient condition for global stability without any interference linearization. This condition will guarantee that, under a designed power control algorithm, the deviations of the power levels in the network from their corresponding optimal values will always remain bounded even when the channel gains change, as long as the variations in the channel gains do not force the network out of its feasibility region.

The concluding remarks are provided in the last section.

7.2 Decision-Feedback versus Information-Feedback Algorithms

Closed-loop power control algorithms can be considered in two categories of *decision-feedback* and *information-feedback* algorithms [31]. In decision-feedback algorithms, the receiver compares the measured SIR with its target value and sends only one or two bits back to the transmitter at the end of every power update interval to command the transmitter to increase or decrease its power level by a fixed or adaptively adjusted step.

On the other hand, in information-feedback algorithms, a real number (e.g., the SIR measurement or the power command) or actually its finely quantized value is sent back to the transmitter. Due to the limitations on the control bandwidth and on the processing time, information-feedback algorithms can run at much slower power update rates than the decision-feedback algorithms. In other words, it is usually unrealistic to expect that information-feedback algorithms mitigate fast fading effects. This is why decision-feedback and information-feedback algorithms have been sometimes characterized in the literature as fast and slow power control algorithms respectively. It should be noted, however, that decision-feedback algorithms do not necessarily result in better overall performance than information-feedback algorithms, even though they can generally run at higher power update rates. In fact, comparing decision-feedback and information-feedback algorithms under various scenarios can be a topic of further research.

We decided to focus on information-feedback algorithms, that is, we assume that the local mean SIR measurements, possibly with some uncertainty, are available at the transmitter. An equivalent scheme would be to assume that the optimal transmit power is calculated at the receiver and is sent back, as a real number, to the transmitter. In the next section, we explore the two main approaches for designing information-feedback power control algorithms.

7.3 SIR Balancing versus SIR Threshold Approaches

An early approach to power control design was *SIR balancing* where the objective was to maximize the minimum SIR of all active users in the network. It was shown in [89] that the optimal SIR and the associated optimal powers could be obtained by solving an eigenvalue problem. Distributed algorithms were later presented in [29] and [90] in order to obtain the optimal powers based on the local information of the users. These algorithms were neglecting the receiver noise and were trying to align the power vector of the network in the direction of an eigenvector. Therefore, they required a normalization procedure in order to avoid drifting all powers to zero or infinity. Unfortunately, some global information was needed to implement such a normalization procedure and this would prevent these algorithms from being fully distributed.

An alternative approach was *SIR threshold* approach, presented in [25], where the objective was for the SIR of each user in the network to be above a desired threshold. It was shown how the optimal powers could be obtained through a simple distributed algorithm. The necessary and sufficient condition for the existence of the optimal powers was expressed as a *feasibility* condition. Various generalizations of this algorithm were later discussed in the literature. A uniform framework along with convergence analysis (under the condition of feasibility) for many of these algorithms were presented in [87].

In this section, we explore these two main approaches and formally show how one can unify them. In unifying the two approaches, we provide some simpler alternative proofs for some of the results.

Consider a cellular system where M users are sharing a single channel. This channel could be a frequency band (FDMA), a time slot (TDMA) or even a spreading code (CDMA). Therefore, for every desired user-base station link, there are $M-1$ interfering links. The received SIR on the uplink channel for user i can now be written as:

$$r_i = \frac{g_{ii}p_i}{\sum_{j\neq i} g_{ij}p_j + \eta_i}, \tag{7.2}$$

where p_i is the transmit power for user i, g_{ii} is the channel gain (or attenuation) from user i to its intended base station (in the linear scale), g_{ij} is the channel gain from user j to the intended base station of user i and η_i is the receiver noise intensity at the intended base station of user i.

Note that even though we decide to focus on the uplink channel, a similar model and similar results can be obtained for the downlink channel. Also note that the above model could similarly be applied when multiple access interference is considered in CDMA systems. In that case, we could consider a single cell scenario where all users are communicating with the same base station and the interference is caused because of the cross correlations among the spreading codes. In that case, g_{ij}'s would incorporate the code cross correlations as well as channel variations due to path loss, shadowing and fast fading. For convenience, we only consider co-channel interference in a multiple cell scenario where users are distributed in different co-channel cells and no adjacent channel interference is assumed. This is shown in Figure 7.1.

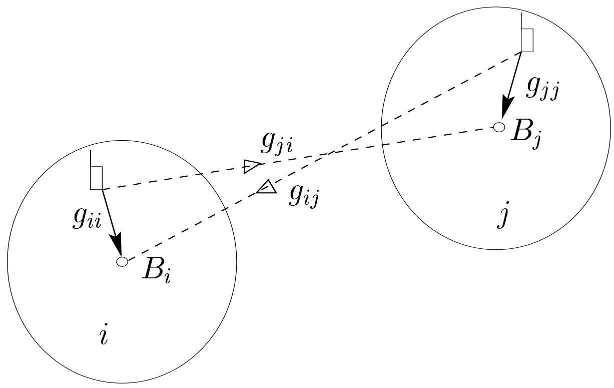


Figure 7.1: Co-channel interference in a multiple cell scenario

Define the normalized channel gain matrix Z as:

$$Z = [z_{ij}], \qquad z_{ij} = \frac{g_{ij}}{g_{ii}}. \tag{7.3}$$

Note that Z is a non-negative stochastic matrix and, in general, is not symmetric. As we shall see, theory of non-negative matrices plays an essential role in both SIR threshold and SIR balancing approaches. Therefore, we start by reviewing some fundamental properties of non-negative matrices [10][41], which are needed in our analysis.

Definition 2. *A non-negative (component-wise) matrix A is called irreducible if $\exists k > 0$ such that $A^k > 0$.*

Lemma 7.3. *A non-negative matrix is irreducible if and only if it has only one* *eigenvalue with maximum modulus and no permutations of its rows and columns can transform it to a block triangular matrix.*

Theorem 5. *Let A be a non-negative irreducible matrix and let $\rho(A)$ be its spectral radius. Then:*

1. *(Perron-Frobenius) $\rho(A) > 0$ is the unique real, positive and simple eigenvalue of A, for which the associated eigenvector can be made positive.*

2. $\rho(A) = \min\{\lambda \,|\exists P \geq 0 : \lambda P \geq AP\}$

Theorem 6. *Let A be a non-negative irreducible matrix and let $\rho(A)$ be its spectral radius. The following statements are equivalent:*

- $\rho(A) < 1$

- *All principal minors of $I - A$ are positive (Metzler).*

- $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ *exists and is positive (component-wise).*

- *For any given non-negative vector u, there exists a non-negative vector P such that $(I - A)P \geq u$.*

Theorem 7. *Let A be any non-negative matrix and let $\rho(A)$ be its spectral radius. Then:*

$$\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij}, \tag{7.4}$$

$$\min_j \sum_i a_{ij} \leq \rho(A) \leq \max_j \sum_i a_{ij}. \tag{7.5}$$

Theorem 8. *Let A be any non-negative matrix and let $\rho(A)$ be its spectral radius. Let $\tilde{A}$ be any principal sub-matrix of A (i.e., any l-by-l sub-matrix formed from l rows and the corresponding (same index) l columns of A). Then $\rho(\tilde{A}) \leq \rho(A)$.*

Note that positive matrices form a subset of non-negative irreducible matrices.

We are now ready to state the SIR balancing result. Note that the normalized channel gain matrix Z is a non-negative matrix. Moreover, as long as we are not considering isolated clusters of users, it can reasonably be assumed to be irreducible.

Theorem 9. (SIR Balancing [89]) *If the receiver noise is neglected for all receivers ($\eta_i = 0,\ \forall i$), then the unique maximum achievable SIR by all users is given by:*

$$\gamma^*_{SB} = \frac{1}{\rho(Z) - 1}, \tag{7.6}$$

where $\rho(Z)$ is the spectral radius of Z, which is also an eigenvalue for Z. The power vector P^* achieving this maximum is the eigenvector of Z associated with $\rho(Z)$.

This SIR balancing theorem is proved in [89] simply by using the results in Theorem 5. Note that:

$$\sum_j z_{ij} = \sum_j \frac{g_{ij}}{g_{ii}} > 1. \tag{7.7}$$

Therefore inequality (7.4) ensures that $\rho(Z) > 1$ and thus $\gamma^* > 0$. Note that the above theorem only specifies the direction of the optimal power vector. In other words, if P^* is optimal, then αP^* will also be optimal for any $\alpha \geq 0$, e.g., $P^* = 0$ is also an optimal power vector! This rather strange result comes from the fact that the receiver noise has been neglected in the model. As we shall see, SIR balancing could provide us with a performance bound in the sense that there exists no other power vector which could yield a higher SIR than γ^* for *all* users in the network. Note that the optimal power vector P^* should be obtained in a centralized form, that is, a central processor needs to collect all the channel gains, form the global matrix Z and calculate the optimal power vector for the network and send back the corresponding optimal power command to every user. As mentioned before, distributed algorithms were presented in the literature [29][90], but they required a normalization procedure to avoid drifting all powers to zero or infinity. Moreover, some global information was needed in this normalization, which would prevent these algorithms from being fully distributed.

On the other hand, in the SIR threshold approach, the objective is for the SIR of every user i to be above a desired threshold γ_i, that is:

$$r_i = \frac{g_{ii}p_i}{\sum_{j\neq i} g_{ij}p_j + \eta_i} \geq \gamma_i. \tag{7.8}$$

It is easy to show that the above constraints can be written in the matrix form as:

$$P \geq \Gamma(Z - I)P + U, \tag{7.9}$$

where $\Gamma = diag(\gamma_1, \ldots, \gamma_M)$ and $U = [u_i] = [\frac{\gamma_i \eta_i}{g_{ii}}]$ and I is the identity matrix. The necessary and sufficient condition for the existence of a positive power vector P, which satisfies the above constraint, is called *feasibility*. In other words, a network of users is called feasible if every user can achieve its desired SIR. The corresponding power vector is then called a feasible power vector. It is clear that feasibility of a network depends on all channel gains and all desired SIRs. In SIR threshold approach, the feasibility condition is quantified and the minimum feasible power vector is obtained.

Theorem 10. (SIR Threshold) *Assuming $U > 0$, a network of users is feasible if and only if $\rho(F) < 1$, where:*

$$F \triangleq \Gamma(Z - I) \Rightarrow f_{ii} = 0, \qquad f_{ij} = \frac{\gamma_i g_{ij}}{g_{ii}}, \quad i \neq j, \tag{7.10}$$

and under the feasibility condition, the optimal power vector is obtained as:

$$P^* = (I - F)^{-1}U. \tag{7.11}$$

The proof of the above theorem directly results from Theorem 6. Note that F is also a non-negative irreducible matrix. The power vector P^* is optimal in the sense that for any other feasible power vector P, we have $P > P^*$.

The above solution for P^* is again a centralized solution. It was shown in [25] that a simple iterative algorithm, which could be implemented in a distributed manner, would converge to P^*. In fact, it is clear that under the condition of feasibility, the optimal power vector P^* is the unique fixed point of the following iterative algorithm:

$$P(n) = FP(n-1) + U, \tag{7.12}$$

and component-wise, we can write:

$$p_i(n) = \frac{\gamma_i}{g_{ii}} \left(\sum_{j \neq i} g_{ij} p_j(n-1) + \eta_i \right) = \frac{\gamma_i}{g_{ii}} I_i(n) = p_i(n-1) \frac{\gamma_i}{r_i(n)}, \tag{7.13}$$

where $I_i(n)$ is the total interference plus noise power at the receiver of the intended base station for user i. Therefore, every user only needs a measurement of its own channel gain and its total interference plus noise in order to update its power level. In fact, at the beginning of the n-th power update period, the local mean channel gain g_{ii} and the local mean total interference plus noise power $I_i(n)$ are measured at the receiver and the new power level $p_i(n)$ is calculated and sent back to the user. Note that $I_i(n)$ depends on the power levels of the users during the $(n-1)$-th power update period. Also no extra delays are assumed for processing and propagation. Moreover, the convergence is proved assuming that all the channel gains and the desired SIRs stay constant for the duration of convergence of the algorithm. This may not always be a reasonable assumption, especially if fast fading is considered while low power update rates are assumed.

Now, our objective is to show how SIR balancing and SIR threshold approaches are related. In other words, we will show how the SIR balancing result could be obtained as a special case in the SIR threshold approach.

Theorem 11. *Assume that all receivers have non-zero noise intensities and all users in the network have the same desired threshold, i.e.,* $\eta_i > 0$, *and* $\gamma_i = \gamma,\ i = 1, 2, \ldots, M$. *Then the network is feasible if and only if:*

$$\gamma < \frac{1}{\rho(Z) - 1} = \gamma^*_{SB}. \tag{7.14}$$

Proof: From Theorem 10 we know that a necessary and sufficient condition for feasibility is $\rho(F) < 1$. Therefore:

$$\rho(F) = \rho(\Gamma(Z - I)) = \rho(\gamma(Z - I)) = \gamma(\rho(Z) - 1) < 1 \Rightarrow \gamma < \frac{1}{\rho(Z) - 1}, \tag{7.15}$$

where we have used the fact that Z is a non-negative irreducible matrix and thus $\rho(Z)$ is itself an eigenvalue of Z and also the fact that $\rho(Z) > 1$.

$\diamondsuit$

The above theorem clearly shows how SIR balancing provides us with a performance bound. The following theorem considers the case where no receiver noise is included.

Theorem 12. *Assume that the receiver noise intensity is zero for all receivers, i.e.,* $U = 0$. *Then* $\gamma = \gamma^*_{SB}$ *is the only SIR threshold, which can be simultaneously achieved by all users in the network using a positive feasible power vector.*

Proof: Assuming $U = 0$ in (7.12), we see that the optimal power vector satisfies $P = FP$. Therefore, a positive feasible power vector exists only if F has an eigenvalue $\lambda_F = 1$, for which the associated eigenvector is positive. We have:

$$F = \Gamma(Z - I) \Rightarrow \lambda_F = \gamma(\lambda_Z - 1). \tag{7.16}$$

But F is a non-negative irreducible matrix. Therefore, from Theorem (5), we know that it has a unique real positive eigenvalue for which the corresponding eigenvector is positive. Moreover, this eigenvalue is equal to its spectral radius $\rho(F)$. Therefore, F will have an eigenvalue at $\lambda_F = 1$ with an associated positive eigenvector if:

$$\gamma = \frac{1}{\rho(Z) - 1} = \gamma^*_{SB}. \tag{7.17}$$

$\diamondsuit$

This theorem shows how the balanced SIR can be obtained as a special case in the SIR threshold approach.

7.4 Network Feasibility

As we mentioned, a network of users is called feasible if all users can achieve their desired SIR thresholds. We saw that if no power constraints are imposed, a necessary and sufficient condition for feasibility is that $\rho(F) < 1$, where matrix F, defined in (7.10), depends on all channel gains and all desired SIR thresholds. In fact, feasibility is equivalent to existence of a positive power vector which satisfies the constraint (7.9). Obviously, imposing power constraints results in more stringent conditions for feasibility.

It can be very helpful if one could quantify the level of feasibility for a network. In fact, it turns out that $\rho(F)$ is an appropriate congestion measure for a network [32]. A network will be more congested if $\rho(F)$ is closer to one. This information can be ultimately used for call admission control purposes.

It is clear that the feasibility level of a network will change as matrix F changes and this can happen under various scenarios. One is when new co-channel users are admitted into the network or some active users are dropped from the network.

The other is when the SIR thresholds for some active users are changed either because of changes in their quality of service requirements or simply through outer loop control commands. An outer loop control scheme may be implemented to adaptively change the SIR threshold for a user according to the measured values of Bit Error Rate or Frame Error Rate. Finally, the feasibility level changes as the channel gains vary due to user motions and fading. The latter two cases indicate the fact that the feasibility condition of a network can change with time even if no user arrivals or departures occur.

In this section, we first provide an insightful sufficient condition for feasibility. We then discuss how user arrival or departures could affect the feasibility and the power levels of the active users in a power-controlled network.

The feasibility condition $\rho(F) < 1$ is a global condition. Therefore, it can be verified only in a centralized manner. Moreover, it gives us little insight about the conditions on the individual channel gains and SIR thresholds in a feasible network. The following theorem provides us with a sufficient condition for feasibility.

Theorem 13. *Consider a network of users with a given channel gain matrix Z and a given SIR threshold matrix Γ. A sufficient condition for feasibility is:*

$$\gamma_i < \frac{g_{ii}}{\sum_{j \neq i} g_{ij}}. \tag{7.18}$$

Proof: Consider matrix F as defined in (7.10). Clearly F is a non-negative matrix for which:

$$\sum_j f_{ij} = \frac{\gamma_i}{g_{ii}} \sum_{j \neq i} g_{ij}. \tag{7.19}$$

From (7.4), it is now clear that a sufficient condition for $\rho(F) < 1$ is:

$$\frac{\gamma_i}{g_{ii}} \sum_{j \neq i} g_{ij} < 1 \Rightarrow \gamma_i < \frac{g_{ii}}{\sum_{j \neq i} g_{ij}}. \tag{7.20}$$

$\diamondsuit$

Note that the above condition is only sufficient. However, it gives us more insight on the relative values of the individual channel gains and desired SIRs in a feasible network. More importantly, it verifies the intuitive fact that higher desired SIR thresholds can be accommodated in a network with a more diagonally dominant channel gain matrix.

As mentioned before, any change in the SIR thresholds for active users can change the feasibility level of a network. One way to quantify the feasibility level is to define a *feasibility margin* or a *feasibility index* [35]:

Definition 3. *Consider a network with a channel gain matrix Z and an SIR threshold matrix Γ. The feasibility margin fm is defined as:*

$$fm \stackrel{\triangle}{=} \min \{x \geq 0 \,|\, x\Gamma \quad \textit{is} \quad \textit{infeasible}\}. \tag{7.21}$$

The notion of feasibility margin is similar to the notion of gain margin in stability theory. It shows how much the active users can simultaneously increase their desired SIR thresholds before making the network infeasible. Using the feasibility condition, one may alternatively write:

$$fm = \min \{x \geq 0 \,|\rho(x\Gamma(Z - I)) \geq 1\}. \tag{7.22}$$

It is then clear that:

$$fm = \frac{1}{\rho(\Gamma(Z - I))} = \frac{1}{\rho(F)}. \tag{7.23}$$

In other words, higher $\rho(F)$ implies lower feasibility margin, which then implies that the network is closer to becoming infeasible. One may also define individual feasibility margins as:

$$fm_i \stackrel{\triangle}{=} \min \{x \geq 0 \,|\, \Gamma_i = diag(\gamma_1, \ldots, \gamma_{i-1}, x\gamma_i, \gamma_{i+1}, \ldots, \gamma_M) \text{ is } \text{ infeasible}\}. \tag{7.24}$$

The feasibility margin for user i, fm_i, shows how much user i can increase its quality of service requirement before making the network infeasible, while other users keep their desired SIRs constant.

While feasibility margin shows how the changes in the SIR thresholds might affect the feasibility level of a network, it is also instructive to quantify the changes in the feasibility and the power levels upon user arrivals and departures.

First, we should note that any new user arrival will decrease the feasibility margin (i.e., increase $\rho(F)$), while any user departure will increase the feasibility margin. This intuitive result can be formally proved using Theorem 8. Now our objective is to quantify the effect of a user arrival or departure. Without any loss of generality, we assume that user M is the user to arrive to or depart from the network. Let F_w be the matrix F with user M in the network and $F_{w/o}$ be the matrix F without the user M. Then:

$$F_w = \begin{bmatrix} F_{w/o} & h_1 \\ \gamma_M h_2^T & 0 \end{bmatrix}, \tag{7.25}$$

where:

$$h_1^T \triangleq \begin{bmatrix} \gamma_1 \frac{g_{1M}}{g_{11}} & \gamma_2 \frac{g_{2M}}{g_{22}} & \dots & \gamma_{M-1} \frac{g_{M-1,M}}{g_{M-1,M-1}} \end{bmatrix}, \tag{7.26}$$

$$h_2^T \triangleq \begin{bmatrix} \frac{g_{M1}}{g_{MM}} & \frac{g_{M2}}{g_{MM}} & \dots & \frac{g_{M,M-1}}{g_{MM}} \end{bmatrix}. \tag{7.27}$$

The maximum achievable SIR along with the required power level for a new user were obtained in [80]. We obtain a more general result, which shows the changes in the power levels of all users in a network when a new user is admitted to the network or when an active user is dropped from the network.

Theorem 14. *Consider a feasible network with $M-1$ users. The maximum achievable SIR for a new user is:*

$$\gamma_{max} = \frac{1}{h_2^T (I - F_{w/o})^{-1} h_1}. \tag{7.28}$$

Proof: For the network to be feasible after the new user arrival, we need to have $\rho(F_w) < 1$. Equivalently, from Theorem 6, all principal minors of $I - F_w$ need to be positive. Since the network is assumed to be feasible before the new user arrival, $I - F_{w/o}$ already has positive principal minors and hence we only need to have $\det(I - F_w) > 0$. We have:

$$\begin{aligned} \det(I - F_w) &= \det \begin{bmatrix} I - F_{w/o} & -h_1 \\ -\gamma_M h_2^T & 1 \end{bmatrix} \\ &= \det(I - F_{w/o}) \left(1 - \gamma_M h_2^T (I - F_{w/o})^{-1} h_1\right) > 0, \end{aligned} \tag{7.29}$$

where we have used the following result for determinants of block matrices:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B) = \det A \det \Delta_A. \tag{7.30}$$

Now since $\det(I - F_{w/o})$ is positive by assumption, in order to have a feasible network after the new user arrival, we need to have:

$$\gamma_M < \frac{1}{h_2^T (I - F_{w/o})^{-1} h_1} \triangleq \gamma_{max}. \tag{7.31}$$

$\diamondsuit$

We shall now compare the optimal power levels of all users with and without user M in the network. We assume that the network is feasible when user M is present. Moreover, if a distributed iterative algorithm is to be implemented, we assume that the channel gains stay constant for the duration of convergence of the algorithm after user M is either admitted to the network or dropped from the network. Using (7.11), we can write:

$$\begin{bmatrix} P_w^* \\ p_M^* \end{bmatrix} = \begin{bmatrix} I - F_{w/o} & -h_1 \\ -\gamma_M h_2^T & 1 \end{bmatrix}^{-1} \begin{bmatrix} U_{1:M-1} \\ u_M \end{bmatrix}, \tag{7.32}$$

where P_w^* is the optimal power vector of all $M-1$ users when user M is present, p_M^* is the optimal power of user M, and:

$$U_{1:M-1}^T \triangleq \begin{bmatrix} \gamma_1 \frac{\eta_1}{g_{11}} & \gamma_2 \frac{\eta_2}{g_{22}} & \dots & \gamma_{M-1} \frac{\eta_{M-1}}{g_{M-1,M-1}} \end{bmatrix}, \tag{7.33}$$

$$u_M \triangleq \gamma_M \frac{\eta_M}{g_{MM}}. \tag{7.34}$$

We use the following lemma to proceed:

Lemma 7.4. (Inverse of a block matrix)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Delta_A^{-1}CA^{-1} & -A^{-1}B\Delta_A^{-1} \\ -\Delta_A^{-1}CA^{-1} & \Delta_A^{-1} \end{bmatrix}, \tag{7.35}$$

where $\Delta_A = D - CA^{-1}B$ *is called the* Schur's complement *of* A *in the block matrix.*

◇

Using the above lemma, we have:

$$\begin{bmatrix} I - F_{w/o} & -h_1 \\ -\gamma_M h_2^T & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} (I-F_{w/o})^{-1} + \frac{\gamma_M (I-F_{w/o})^{-1} h_1 h_2^T (I-F_{w/o})^{-1}}{1-\gamma_M h_2^T (I-F_{w/o})^{-1} h_1} & \frac{(I-F_{w/o})^{-1} h_1}{1-\gamma_M h_2^T (I-F_{w/o})^{-1} h_1} \\ \frac{\gamma_M h_2^T (I-F_{w/o})^{-1}}{1-\gamma_M h_2^T (I-F_{w/o})^{-1} h_1} & \frac{1}{1-\gamma_M h_2^T (I-F_{w/o})^{-1} h_1} \end{bmatrix} \tag{7.36}$$

Now define:

$$P_{w/o}^* \triangleq (I - F_{w/o})^{-1} U_{1:M-1}, \tag{7.37}$$

$$I_{M,w/o} \triangleq g_{MM} h_2^T P_{w/o}^* + \eta_M. \tag{7.38}$$

Note that $P_{w/o}^*$ is the optimal power vector of all $M-1$ users when user M is not present in the network. Also $I_{M,w/o}$ is the total interference plus noise at the intended base station of user M before it is admitted or after it is dropped. Using (7.32), (7.36), (7.37) and (7.38), and the definition for γ_{max} in (7.28), it is straightforward to show:

$$\begin{aligned} P_w^* &= P_{w/o}^* + p_M^*(I - F_{w/o})^{-1}h_1, && (7.39) \\ p_M^* &= \frac{I_{M,w/o}}{g_{MM}} \frac{\gamma_M}{1 - \frac{\gamma_M}{\gamma_{max}}}. && (7.40) \end{aligned}$$

Moreover, since:

$$r_M = \frac{g_{MM}p_M^*}{I_M} = \gamma_M, \tag{7.41}$$

we have:

$$I_M = \frac{I_{M,w/o}}{1 - \frac{\gamma_M}{\gamma_{max}}}. \tag{7.42}$$

where I_M is the interference plus noise level at the intended base station of user M when it is present in the network.

Interesting conclusions can be made from the above equations. Let user M be a new user, which can be admitted to the network, that is, $\gamma_M < \gamma_{max}$. The optimal power level for this new user will be directly proportional to the interference plus noise level at its intended base station before getting into the network. Also the optimal power level for the new user will increase as its desired SIR gets closer to γ_{max}. In fact, as $\gamma_M \to \gamma_{max}$, all power levels in the network go to infinity. Obviously, a new user will be able to achieve lower SIR levels if any power constraints are imposed.

Equation (7.39) quantifies the changes in the power levels of all users in the network when a new user is admitted or when an active user is dropped. It is clear that when a user is admitted (dropped), the increase (decrease) in the optimal power levels of all active users is directly proportional to the optimal power level of the new (dropped) user. Moreover, the change in the power level of any active user linearly depends on h_1, which includes all desired SIRs, all the channel gains from all active users to their intended base stations and all the channel gains from the new (dropped) user to the intended base stations of all other active users. Finally, equation (7.42) shows how the changes in the power levels of active users translate to the change in the interference plus noise level at the intended base station of the new (dropped) user.

7.5 Power Control as a Decentralized Regulation Problem

So far, we have discussed SIR balancing and SIR threshold as the two main approaches in power control design. While trying to unify the two approaches, we reviewed how a simple distributed iterative algorithm (7.13) may be implemented for power control. This algorithm is actually a simple integrator algorithm in the logarithmic scale. This fact has recently initiated a new approach for power control design using concepts from control theory. This approach could be specifically helpful if models for the channel gain variations are to be incorporated in the design. In this section, we discuss how power control can be posed as a decentralized regulator problem.

Using a bar on the variables to indicate the values in dB, we can write the distributed algorithm in (7.13) in logarithmic scale as:

$$\bar{p}_i(n) = \bar{p}_i(n-1) + (\bar{\gamma}_i - \bar{r}_i(n)) \stackrel{\triangle}{=} \bar{p}_i(n-1) + \bar{e}_i(n), \tag{7.43}$$

where $\bar{p}_i(n)$ is the power level in dBm for user i for the duration of the n-th power update period and $\bar{r}_i(n)$ is the SIR in dB for the same user at the beginning of the n-th power update period:

$$\bar{r}_i(n) = \bar{p}_i(n-1) + \bar{g}_{ii}(n) - \bar{I}_i(n). \tag{7.44}$$

Moreover, $\bar{I}_i(n)$ is the local mean interference plus noise power in dBm available

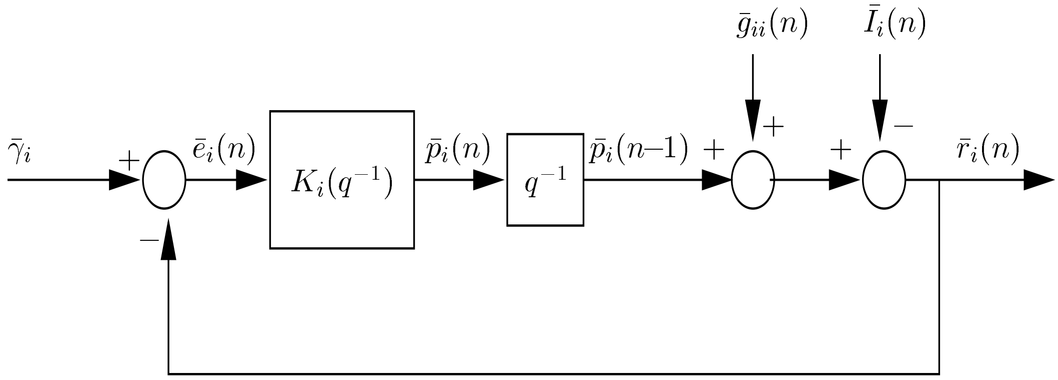


Figure 7.2: A local power control loop, associated with a single user

at the beginning of the n-th power update period:

$$\bar{I}_i(n) = 10\log_{10}\left(\sum_{j\neq i} g_{ij}10^{\frac{\bar{p}_j(n-1)}{10}} + \eta_i\right). \tag{7.45}$$

It is now easy to see that this algorithm is, in fact, a simple unity gain integrator algorithm in a closed local loop, as shown in Figure 7.2. The controller transfer function in this case is:

$$K_i(q^{-1}) = \frac{\bar{P}_i(q^{-1})}{\bar{E}_i(q^{-1})} = \frac{1}{1-q^{-1}}, \tag{7.46}$$

where q is the shift operator. Therefore, the network can be seen as a set of interconnected local loops, each of which is associated with a single user. It should be realized that the couplings among the local loops is through the interference function (7.45), which, in general, is nonlinear. The decentralized regulator formulation of the power control problem can now be presented as:

- Design a set of local controllers $K_i(q^{-1})$ such that the SIR for every user, $\bar{r}_i$, tracks a desired threshold $\bar{\gamma}_i$ with a certain performance while the global network remains stable.

This approach has already initiated research on using control theory concepts for power control design [21, 31, 68]. Note that the local loops in Figure 7.2 are quite general and can be modified to accommodate different modeling assumptions. For example, extra delay blocks may be inserted in the feedback path to model processing and propagation delays. In fact, one step delay is typically assumed when high power update rates are considered [79]. As another example, a saturation block may be inserted in the forward path after the controller to model the maximum and minimum power constraints. It should also be mentioned that we have implicitly assumed a linear time invariant controller by writing $K_i(q^{-1})$. However, in general, the controller itself can be a nonlinear block, as is the case for *Fixed-Step* power control algorithms.

This approach can potentially open a new frontier for power control design, where advanced techniques from control theory can be used to design power control algorithms under various constraints, while taking into account available models for channel gain variations. Moreover, joint design of inner and outer loop control algorithms may be facilitated in this approach. One challenge, however, is to deal with many practical implementation constraints, such as control bandwidth limitation, computational cost, etc. The other challenge is to design decentralized controllers. In fact, as we have seen throughout the previous parts of this thesis, there are many fundamental open issues in designing decentralized controllers, which have recently attracted a lot of attention in the control community. It could be very difficult to achieve a prescribed level of performance while ensuring global stability of the network.

7.6 Global Stability of the Network

Our objective in this section is to address the issue of global stability. Unfortunately, stability and convergence of power control algorithms, designed as decentralized regulators, cannot be verified through simple techniques such as the one presented in [87]. A robust control framework was proposed in [31] to obtain a sufficient condition for global stability using a linearized interference function. We will use a similar approach, but with a different notion for stability, and we obtain a more general sufficient condition for global stability without any interference linearization. We also mention how network feasibility is addressed in this approach.

We consider a system to be stable if bounded inputs generate bounded outputs. In robust control terminology [18][91], we use ℓ_∞ norm to quantify the size of the signals in the system and ℓ_∞-induced norms to quantify the amplification of the signals, i.e., the size of operators or transfer functions.

Definition 4. *Let x be an n-dimensional real-valued discrete signal. The ℓ_∞ norm is defined as the maximum amplitude that any component of the signal attains over all time, that is:*

$$||x||_\infty \triangleq \sup_k \max_i |x_i(k)|. \tag{7.47}$$

The space of all n-dimensional real-valued discrete signals with finite ℓ_∞ norm is denoted by ℓ_∞^n. Let T be an operator from one normed linear space to another. The ℓ_∞-induced norm of T is defined as:

$$||T||_{\ell_\infty - induced} \triangleq \sup_{||x||_\infty \leq 1} ||Tx||_\infty. \tag{7.48}$$

Definition 5. (Closed-Loop Stability) *Consider the feedback loop in Figure 7.3. The closed-loop system is ℓ_∞-stable if $||G_{cl}(G_1, G_2)||_{\ell_\infty - induced} < \infty$, where $G_{cl}(G_1, G_2)$ is the closed-loop operator:*

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = G_{cl}(G_1, G_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{7.49}$$

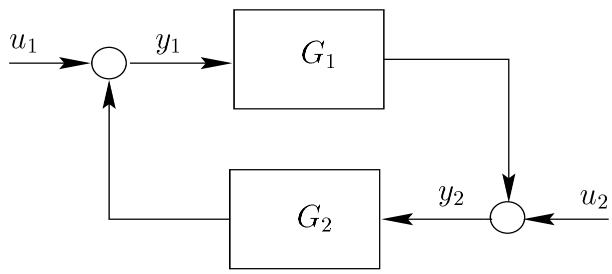


Figure 7.3: Closed-Loop Stability

We will obtain a sufficient global stability condition using a fundamental stability result called the *Small Gain Theorem*:

Theorem 15. (Small Gain Theorem) *Consider the feedback loop in Figure 7.3. Let $G_1 : \ell_\infty^n \to \ell_\infty^m$ and $G_2 : \ell_\infty^m \to \ell_\infty^n$ be two stable operators, and assume that the closed loop system is well-posed (i.e., for any $u_1, u_2 \in \ell_\infty$, there exists a unique solution for $y_1, y_2 \in \ell_\infty$). Then the closed-loop system is stable if* $||G_1||_{\ell_\infty - induced}||G_2||_{\ell_\infty - induced} < 1$.

Note that the above theorem only states a sufficient condition, which may be conservative in some cases.

As we mentioned, a network of co-channel users can be seen as a nonlinearly coupled set of local loops. In fact the global network can be depicted as in Figure 7.4, where $G(q^{-1})$ is a block diagonal closed-loop transfer function matrix from interference $\bar{I}(n)$ to power $\bar{P}(n-1)$ and $\bar{I}(.)$ is a nonlinear operator, which produces interference plus noise in dBm from the power levels. Note that $G_i(q^{-1})$ is also equal to the closed-loop transfer function from $\bar{\gamma}_i$ to $\bar{r}_i$.

We have:

$$\bar{P}(n-1) = G(q^{-1})(\bar{I}(n) - \bar{g}(n) + \bar{\gamma}(n)), \tag{7.50}$$

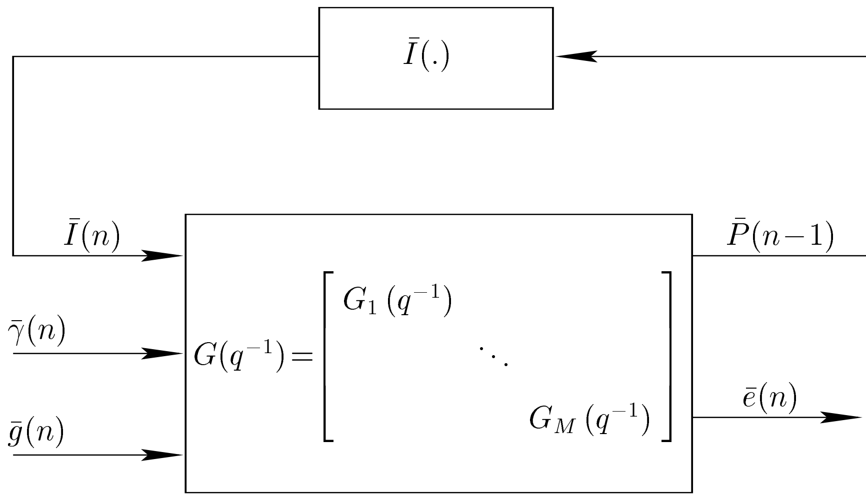


Figure 7.4: The Power-Controlled Global Network

where:

$$\bar{g} \triangleq [\bar{g}_{11} \dots \bar{g}_{MM}]^T, \tag{7.51}$$

$$\bar{\gamma} \triangleq [\bar{\gamma}_1 \dots \bar{\gamma}_M]^T, \tag{7.52}$$

$$\bar{I}(n) \triangleq \left[\bar{I}_1(\bar{P}(n-1)) \dots \bar{I}_M(\bar{P}(n-1))\right]^T. \tag{7.53}$$

Now assume that the network always stays feasible. Note that we are not assuming channel gains to be constant. But we only assume that the time variations of the channel gains do not push the network out of its feasibility region. Therefore, at any instant of time, there exists an instantaneous bounded optimal power vector $\bar{P}^*$, which is related to the corresponding optimal interference as:

$$\bar{P}^*(n-1) = \bar{I}^*(n) - \bar{g}(n) + \bar{\gamma}(n). \tag{7.54}$$

Since we are not considering user arrival or departures, $\bar{P}^*$ will be constant as long as the desired SIR thresholds and the channel gains remain constant. We now consider the deviations of the power and interference levels in the network,

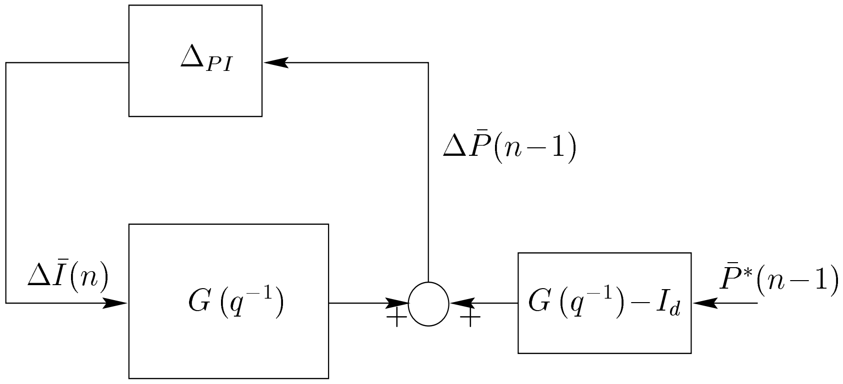


Figure 7.5: The Power-Controlled Global Network in a Variational Form

at every instant of time, relative to their optimal values, that is:

$$\Delta\bar{P} \triangleq \bar{P} - \bar{P}^{*}, \tag{7.55}$$

$$\Delta\bar{I} \triangleq \bar{I} - \bar{I}^{*}. \tag{7.56}$$

Using (7.50) and (7.54), we can now write:

$$\bar{P}(n-1) - G(q^{-1})\bar{P}^{*}(n-1) = G(q^{-1})\Delta\bar{I}(n). \tag{7.57}$$

Hence:

$$\begin{aligned} \Delta\bar{P}(n-1) &= \bar{P}(n-1) - \bar{P}^{*}(n-1) \\ &= \bar{P}(n-1) - G(q^{-1})\bar{P}^{*}(n-1) + \left(G(q^{-1}) - I_d\right)\bar{P}^{*}(n-1) \\ &= G(q^{-1})\Delta\bar{I}(n) + \left(G(q^{-1}) - I_d\right)\bar{P}^{*}(n-1), \end{aligned} \tag{7.58}$$

where I_d is the identity matrix. The network can then be shown as in Figure 7.5, where Δ_{PI} is the nonlinear operator transforming $\Delta\bar{P}$ to $\Delta\bar{I}$. We can show that Δ_{PI} is a contractive operator in the sense that $||\Delta_{PI}||_{\ell_\infty-induced} < 1$. To do so, we need to use the *Mean Value Theorem* [48]. We prove our result after reviewing this theorem.

Theorem 16. (Mean Value Theorem) *Assume that $f : \Re^n \to \Re$ is a continuously differentiable function at every point of an open set containing the line segment L joining two vectors x and y in $\Re^n$. Then there exists a point z on L such that:*

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x). \tag{7.59}$$

Lemma 7.5.

$$||\Delta_{PI}||_{\ell_\infty - induced} < 1. \tag{7.60}$$

Proof: Using (7.45), it is straightforward to show:

$$\frac{\partial \bar{I}_i}{\partial \bar{p}_j} = \begin{cases} 0 & i = j \\ \frac{g_{ij} p_j}{\sum_{k \neq i} g_{ik} p_k + \eta_i} & i \neq j \end{cases}. \tag{7.61}$$

Remember again that the variables without bar indicate values in linear scale. From the Mean Value Theorem, we know that for every i and for every optimal power vector $\bar{P}^*$, there exists a power vector $\tilde{\bar{P}}$ lying on the line segment between $\bar{P}$ and $\bar{P}^*$ such that:

$$\Delta \bar{I}_i = \left.\frac{\partial \bar{I}_i}{\partial \bar{P}}\right|_{\bar{P} = \tilde{\bar{P}}} \Delta \bar{P}. \tag{7.62}$$

Now using (7.61) and (7.62) and assuming $||\Delta \bar{P}||_\infty \leq 1$, which then implies $|\Delta \bar{p}_i(k)| \leq 1$ for all $i = 1, \ldots, M$ and $k = 0, 1, \ldots$, we can write:

$$\begin{aligned} |\Delta \bar{I}_i(k)| &= \left| \sum_{j \neq i} \frac{g_{ij}(k) \tilde{p}_j(k-1)}{\sum_{l \neq i} g_{il}(k) \tilde{p}_l(k-1) + \eta_i} \Delta \bar{p}_j(k-1) \right| && (7.63) \\ &\leq \sum_{j \neq i} \frac{g_{ij}(k) \tilde{p}_j(k-1)}{\sum_{l \neq i} g_{il}(k) \tilde{p}_l(k-1) + \eta_i} |\Delta \bar{p}_j(k-1)| && (7.64) \\ &\leq \sum_{j \neq i} \frac{g_{ij}(k) \tilde{p}_j(k-1)}{\sum_{l \neq i} g_{il}(k) \tilde{p}_l(k-1) + \eta_i} < 1. && (7.65) \end{aligned}$$

Therefore:

$$||\Delta \bar{I}||_\infty = \sup_k \max_i \left| \Delta \bar{I}_i(k) \right| < 1, \tag{7.66}$$

and hence:

$$||\Delta_{PI}||_{\ell_\infty - induced} = \sup_{||\Delta \bar{P}||_\infty \leq 1} ||\Delta \bar{I}||_\infty < 1 \tag{7.67}$$

Note that $||\Delta_{PI}||_{\ell_\infty - induced} = 1$ if no receiver noise is considered for any of the receivers.

$\diamondsuit$

It is clear that stability of every local loop is a necessary (but not sufficient) condition for global stability. We are now ready to state a sufficient condition for global stability of the network on a single channel.

Theorem 17. (Global Stability) *Consider the network in Figure 7.5. Assume that the network is always feasible, i.e., there always exists a bounded power vector P^* satisfying (7.54). Then the network is globally stable if for every user i:*

$$\left|\left|G_i(q^{-1})\right|\right|_{\ell_\infty - induced} \leq 1. \tag{7.68}$$

Proof: Since $G_i(q^{-1})$ always incorporates a delay, it is easy to see that the operator $\Delta_{PI}G$ is always strictly causal and hence the closed loop system in Figure 7.5 is always well-posed. Moreover, the feasibility assumption guarantees the existence of a bounded P^*. Therefore, if $||G_i(q^{-1})||_{\ell_\infty - induced} \leq 1$ for every user i, we will have $||G(q^{-1})||_{\ell_\infty - induced} \leq 1$, and using Lemma Lemma 7.5, the global stability of the network will be established simply by invoking the Small Gain Theorem.

$\diamondsuit$

It should be mentioned that for Single Input Single Output cases, the $\ell_\infty - induced$ norm of the system is equal to the ℓ_1 norm of the impulse response sequence of the system. Therefore:

$$\left|\left|G_i(q^{-1})\right|\right|_{\ell_\infty-induced} = ||g_i||_1 = \sum_{k=0}^{\infty} |g_i(k)|\,, \tag{7.69}$$

where g_i denotes the impulse response associated with the transfer function G_i. The above theorem states that if the feasibility condition is not violated and if (7.68) is satisfied, then the deviations of the power levels of all the users in the network from their corresponding optimal values will always remain bounded. Even though the condition (7.68) is only sufficient, and might be conservative in some cases, it can still help us design new stable algorithms and analyze the stability of current algorithms under channel gain variations. We will show this by an example.

But first, we want to compare our result with the one presented in [31]. It was shown in [31] that if the channel gains stay constant and if the network is feasible (i.e., a constant optimal power vector exists), and if the interference function is linearized around this optimal power vector, then a sufficient condition for global stability of the linearized network (in the $\ell_2 - induced$ norm sense) is:

$$\left|\left|G_i(q^{-1})\right|\right|_{\ell_2-induced} = \sup_{\omega} \left|G_i\left(e^{j\omega}\right)\right| \leq 1. \tag{7.70}$$

This means that if the power vector of the network deviates a little bit from the optimal power vector, and as long as all the channel gains stay constant, the power levels will asymptotically move back to their optimal values. In contrast, in deriving the sufficient condition (7.68), no constant channel gain assumption was made and no linearization was involved. However, the stability in $\ell_\infty - induced$ norm does not imply asymptotic convergence of the power level deviations to zero. Instead, it implies that the deviations always remain bounded even if the optimal power vector changes due to the variations in the channel gains. Also

(7.68) is sometimes more conservative, since we always have:

$$||G_i(q^{-1})||_{\ell_2-induced} \leq ||G_i(q^{-1})||_{\ell_\infty-induced}. \tag{7.71}$$

Example: Consider the integral algorithm in (7.43) with gain β, i.e.,:

$$\bar{p}_i(k) = \bar{p}_i(k-1) + \beta(\bar{\gamma}_i - \bar{r}_i(k)), \tag{7.72}$$

or in linear scale:

$$p_i(k) = p_i(k-1)\left(\frac{\gamma_i}{r_i(k)}\right)^{\beta}. \tag{7.73}$$

We have:

$$G_i(q^{-1}) = \frac{q^{-1}K_i(q^{-1})}{1+q^{-1}K_i(q^{-1})} = \frac{\beta q^{-1}}{1-(1-\beta)q^{-1}}. \tag{7.74}$$

We should first note that for the local loops to be stable we need to have $\beta \in (0, 2)$. It is now easy to show that for $0 \leq \beta \leq 1$, we have:

$$||G_i(q^{-1})||_{\ell_2-induced} = ||G_i(q^{-1})||_{\ell_\infty-induced} = 1.0\ , \tag{7.75}$$

and when β becomes larger than one, both induced norms start increasing. This proves that not only do the power levels, obtained from the distributed iterative algorithm in [25] (where $\beta = 1$ is assumed), converge to their optimal levels if the channel gains stay constant, but also, under the channel gain variations, the deviations of the power levels from their optimal values always remain bounded.

It is instructive to also consider the case where an additional delay is assumed for processing and propagation, i.e., one step delay is inserted in the feedback path in Figure 7.2. In this case:

$$G_i(q^{-1}) = \frac{q^{-2}K_i(q^{-1})}{1+q^{-2}K_i(q^{-1})} = \frac{\beta q^{-2}}{1-q^{-1}+\beta q^{-2}}. \tag{7.76}$$

First note that $\beta = 1$ will result in closed-loop poles on the unit circle and therefore instability of the local loops. The $\ell_\infty - induced$ and $\ell_2 - induced$ norms

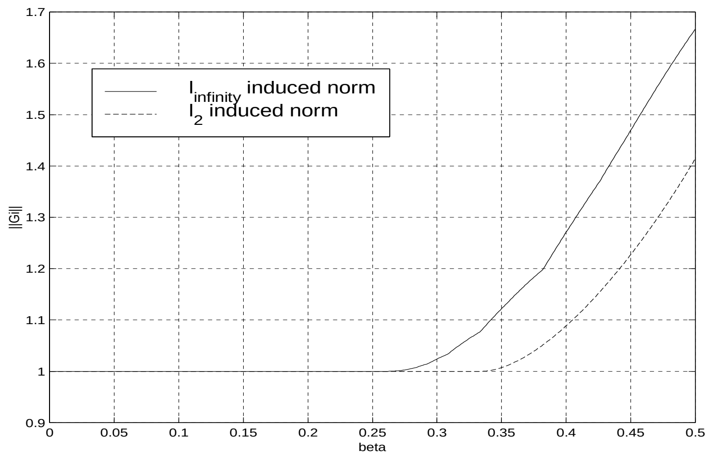


Figure 7.6: $\ell_\infty - induced$ and $\ell_2 - induced$ norms for G_i in the one step delayed case

of G_i are shown in Figure 7.6. It can be seen that in order to guarantee the bounded deviations of the power levels in the network (i.e., the global stability in the ℓ_∞ sense), we need to approximately have $\beta < .27$. Moreover, to ensure the global stability of the linearized system in the ℓ_2 sense, we need to have $\beta < 0.33$. These bounds on the gain are rather small and could therefore result in slow responses to the changes in the SIR thresholds or the channel gains. However, remember that the sufficient conditions for global stability have been obtained under worst case scenarios and therefore might yield conservative requirements in some cases.

$\diamondsuit$

7.7 Summary

We reviewed SIR balancing and SIR threshold as the two main approaches for power control in cellular wireless systems. We then tried to unify the two approaches by formulating the SIR balancing approach as a special case in the SIR threshold approach. We showed how the balanced SIR could be obtained as a performance bound for a network where all the users want to achieve the same SIR. Then we focused on the feasibility condition for a network. We obtained an insightful sufficient condition for feasibility, which formally showed the fact that higher SIR levels can be supported in a network with a more diagonally dominant channel gain matrix. We then discussed how the feasibility and the power levels in a network change when new users are admitted to the network or active users are dropped. We proved that the increase (decrease) in the optimal power level of all active users in the network is linearly proportional to the optimal power level of the new (dropped) user. Then we reviewed the decentralized regulator formulation for power control problem. Using a robust control framework, we obtained a sufficient condition, which would guarantee that the deviations of the power levels from their corresponding optimal values always remain bounded. We then showed that if no extra delay is considered for processing and propagation, the widely proposed integrator algorithm does indeed yield a globally stable network as long as the variations of the channel gains do not force the network out of its feasibility region. As future work, one could try to actually quantify some bounds on the power level deviations.

Moreover, designing better power control algorithms, which can incorporate additional information about the channel gain variations, can be a topic of further research. In fact, in the next chapter, we propose a novel predictive power control algorithm.

CHAPTER 8

A Predictive Power Control Algorithm

8.1 Introduction

In most power control algorithms that have been proposed in the literature, the channel gains are assumed to be constant for the duration of the convergence of the algorithm, and therefore no fading effects are considered. It was recently shown in [46] that the optimal powers obtained from the SIR balancing approach, under constant gain assumptions, are very close to the the optimal powers that minimize the Rayleigh fading induced outage probability for every link. Due to the limitations on the control bandwidth and the computational cost, information-feedback algorithms usually assume long power update periods (in the order of few hundred milliseconds), and are designed to track only the slow channel variations. On the other hand, fast power control algorithms have been developed where only a single bit feedback is employed to command the users to increase or decrease their power levels by a fixed or adaptively adjusted step. The power update rate for these algorithms can be on the order of few hundred hertz. Therefore, they may have some effect on the fast fading in the channel.

Some researchers have tried to analyze and possibly modify the power control algorithms to take into account the channel gain variations and the fading induced measurement errors. In [2] it was shown how the desired SIR for the users may be scaled up to guard against the user mobility effects. In [75] a simulation study was performed to investigate the user mobility effects on a slow integrator power control algorithm. In [52] a modification of the distributed SIR balancing algorithm was proposed, which was less sensitive to SIR measurement errors. Also in [76] stochastic measurements were incorporated in the power control algorithm, and it was shown that the power levels converge, in the mean square sense, to the optimal power levels. More recently, it was shown in [51] how a simple Kalman Filter may be designed to smooth out the interference measurements. Also in [31] it was mentioned how a minimum-variance power control algorithm may be designed when the channel gain variations are modeled by filtered white noise sequences.

Our objective in this chapter is to design a distributed predictive power control algorithm. We try to obtain accurate enough models for the slow variations in the channel gains and the interference powers. We then design simple Kalman filters for every user to obtain the one-step predicted values for both the interference level and the user's channel gain from its intended base station. We try to tune the filters for a typical mobile radio environment and then conjecture, and show through simulations, in the next chapter, that the filters are indeed robust under a broad range of parameters such as user velocities and shadowing correlation distances. The predicted measurements from the Kalman filters are then incorporated in an integrator algorithm to update the power levels.

This chapter is organized as follows. In Section 8.2 we propose a first order white noise driven Markov sequence on top of a constant bias to model the slow variations in the channel gains and the interference levels. In Section 8.3 we explain how simple Kalman filters can be designed for optimal one-step prediction of the channel gains and the interference levels. We also mention how the filter parameters may be tuned for a typical mobile radio environment. In Section 8.4 we show that the sufficient conditions for global stability are satisfied when the Kalman filters are incorporated in the power control loops. Concluding remarks are given in the final section.

8.2 Models for Channel Gain and Interference Variations

The variations in the channel gains can be characterized by the slowly changing shadow fading and the fast multipath fading on top of the distance loss. The first-order statistics of shadow fading, in the logarithmic scale, is usually represented with a Gaussian distribution, whose mean is monotonically decreasing with the mobile to base station distance. Its standard deviation σ_s depends on the environment and is reported to range from 4 to 12 dB with 8 dB as the typical value for urban macro-cellular environments (with cell radii of about 1 km or more and base antenna heights of 30m or more) [43][72]. As for the spatial (or temporal) correlation in shadow fading, we use the simple first-order Markov model presented in [30].

The channel gain from every user i to its intended base station, in the logarithmic scale, is therefore modeled as:

$$\begin{aligned} \bar{g}_{ii}(n) &= \bar{g}_{ii}^0 + \delta\bar{g}_{ii}(n) && (8.1) \\ \delta\bar{g}_{ii}(n) &= a\delta\bar{g}_{ii}(n-1) + w_g(n-1), && (8.2) \end{aligned}$$

where $\bar{g}_{ii}^0$ is a constant bias and w_g is a zero mean white Gaussian noise sequence. The constant bias accounts for the antenna gains and the distance loss in the filter. The parameter a is obtained as:

$$a = e^{-\frac{vT}{X_s}}, \tag{8.3}$$

where v is the user velocity and T is the update period. Note that vT is the distance that the user moves during one update period. Moreover X_s is called the shadowing *correlation distance*. It is the distance at which the normalized correlation decreases to e^{-1}. To see this, note that the autocorrelation function for $\delta\bar{g}$ can be obtained as:

$$R_{\delta\bar{g}}(m) \triangleq \mathrm{E}\left[\delta\bar{g}(m+n)\delta\bar{g}(n)\right] = \frac{\sigma_{w_g}^2}{1-a^2}a^{|m|} = \sigma_s^2 a^{|m|}, \tag{8.4}$$

where σ_{w_g} denotes the standard deviation of the noise sequence w_g. Note that, given the standard deviation for shadowing σ_s and the value for a, the standard deviation for the driving white noise sequence can be obtained.

In order to design distributed algorithms, we need to decouple the local loops in the network. For this purpose, the interference plus noise should be modeled independently for every user. One approach is to treat interference plus noise simply as a bounded disturbance for every user and design the power control algorithm based on worst case considerations. However, we decide to model the interference plus noise, similar to the channel gains, by white noise driven first-order Markov variations on top of a constant bias. That is:

$$\bar{I}_i(n) = \bar{I}_i^0 + \delta\bar{I}_i(n) \tag{8.5}$$

$$\delta\bar{I}_i(n) = a\delta\bar{I}_i(n-1) + w_I(n-1), \tag{8.6}$$

where w_I is a zero-mean white Gaussian noise sequence independent of w_g, but with the same variance. In fact, the same parameters are used in both models for channel gains and interference levels. While this model may not exactly capture the slow variations in the interference in a power-controlled system, it can still be reasonable when such slow fluctuations in the interference levels are dominated by shadow fading. We use this model in a Kalman Filter to obtain the one-step predicted measurements of the local mean interference values.

The power update period should now be selected such that the fast multipath fluctuations are averaged out while the slower shadowing fluctuations are being tracked. It was shown in [27] that, under the flat Rayleigh fading assumption, when a first order low-pass filter or simply a moving average filter is used to obtain the local mean values of the measurements, the averaging error in dB will have a Gaussian distribution, whose mean can be made zero by appropriate choice of the filter DC gain and whose standard deviation depends on the shadow fading standard deviation σ_s, the ratio of the shadowing correlation distance to the carrier wavelength X_s/λ, and the normalized measurement time $f_m T$, where $f_m = v/\lambda$ is the maximum Doppler frequency.

It is now clear that the model parameters not only depend on the environment through the values of the shadowing standard deviation and the shadowing correlation distance, but also depend on the user velocity. While one can think of implementing individual *adaptive* Kalman filters for each user, where the model parameters are continuously updated based on the available information about the user velocities, we choose to consider a fixed model to design and implement the same filters for all the users in the network. There are two main reasons for this. One is that for a rather broad range of user velocities, the values for a and σ_{w_g}, and as shown in [27], the averaging error variance only slightly change and we believe that the Kalman filters will be robust to such changes. The other reason is that while some techniques have been already proposed for user velocity estimation in mobile environments (refer to [57] and the references therein), most of them fail to provide accurate estimates in real time.

8.3 Kalman Filter Design

Using a set of available measurements, corrupted with Gaussian noise, a Kalman filter recursively obtains the minimum mean squared error estimates of a set of variables that are varying according to a given dynamic model. Kalman filters have proved to be strong estimation tools in a very wide range of applications [69]. As examples of applications in communication systems, Kalman filters have been used for channel equalization [60], interference estimation for call admission in CDMA networks [20] and for power control in packet-switched broadband TDMA networks [52].

We propose a predictive power control algorithm, where two Kalman filters are employed to provide the one-step predicted estimates of both the channel gains and the interference levels for every user, which are then used in an integrator algorithm to update the power levels. Using (8.1) and (8.2) for the channel gains, we can write:

$$\bar{g}_{ii}(n) = a\bar{g}_{ii}(n-1) + (1-a)\bar{g}_{ii}^{0} + w_g(n-1). \tag{8.7}$$

Similarly, using (8.5) and (8.6) for the interference levels, we can write:

$$\bar{I}_i(n) = a\bar{I}_i(n-1) + (1-a)\bar{I}_i^{0} + w_I(n-1). \tag{8.8}$$

The idea is to design two simple Kalman filters that use the erroneous local mean measurements, available to every user, to estimate the constant biases in the models and provide the one-step predicted estimates of the channel gains and the interference levels. As mentioned, the same models are used for all the mobiles in the network. Hence we eliminate the indices i and ii for a simpler notation.

It is now appropriate to represent both models in the state-space form. Let us define:

$$x_{g1}(n) \triangleq \bar{g}(n), \qquad x_{g2}(n) \triangleq \bar{g}^0, \tag{8.9}$$

$$x_{I1}(n) \triangleq \bar{I}(n), \qquad x_{I2}(n) \triangleq \bar{I}^0. \tag{8.10}$$

The state-space models can then be obtained as:

$$x_g(n) \triangleq \begin{bmatrix} x_{g1}(n) \\ x_{g2}(n) \end{bmatrix} = \begin{bmatrix} a & 1-a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{g1}(n-1) \\ x_{g2}(n-1) \end{bmatrix} + \begin{bmatrix} w_g(n-1) \\ w_{g0}(n-1) \end{bmatrix}, \tag{8.11}$$

$$x_I(n) \triangleq \begin{bmatrix} x_{I1}(n) \\ x_{I2}(n) \end{bmatrix} = \begin{bmatrix} a & 1-a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{I1}(n-1) \\ x_{I2}(n-1) \end{bmatrix} + \begin{bmatrix} w_I(n-1) \\ w_{I0}(n-1) \end{bmatrix}. \tag{8.12}$$

where w_{g0} and w_{I0} are two mutually independent *fictitious* zero mean white Gaussian noise sequences , which are also independent from w_g and w_I. They are required to prevent the Kalman filters from relying too much on their estimates of $\bar{g}^0$ and $\bar{I}^0$ and to force them to take into account the new measurements. This makes the filters robust to the uncertainties in the models and enables them to deal with the changes in $\bar{g}^0$ (e.g., due to path loss variations) or $\bar{I}^0$ (e.g., due to user arrivals or departures).

The measurement equations for the filter can now be written as:

$$y_g(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{g1}(n) \\ x_{g2}(n) \end{bmatrix} + v_g(n), \tag{8.13}$$

$$y_I(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{I1}(n) \\ x_{I2}(n) \end{bmatrix} + v_I(n), \tag{8.14}$$

where y_g and y_I respectively denote the measured local mean values of the channel gain and the interference level, and v_g and v_I are mutually independent zero mean white Gaussian noise sequences, which are assumed to be independent from all other noise sequences in the model and are used to model the fast fading induced averaging errors and other possible uncertainties in the local mean measurements. Remember that all the variables are expressed in a logarithmic scale. For a simpler notation, define:

$$\begin{aligned} A_f &\triangleq \begin{bmatrix} a & 1-a \\ 0 & 1 \end{bmatrix}, & (8.15) \\ H_f &\triangleq \begin{bmatrix} 1 & 0 \end{bmatrix}. & (8.16) \end{aligned}$$

Now starting from initial estimates $\hat{x}_g(0)^-$ and $\hat{x}_I(0)^-$, the *measurement update* equations for the filters are expressed as:

$$\begin{aligned} \hat{x}_g(n)^+ &= \hat{x}_g(n)^- + L_g(n)\left(y_g(n) - H_f\hat{x}_g(n)^-\right), & (8.17) \\ \hat{x}_I(n)^+ &= \hat{x}_I(n)^- + L_I(n)\left(y_I(n) - H_f\hat{x}_I(n)^-\right), & (8.18) \end{aligned}$$

where $\hat{x}_g(n)^-$ and $\hat{x}_I(n)^-$ respectively denote the *propagated* (*a priori*) estimates of the channel gain and the interference level at the end of the $(n-1)$-th power update period. Hence, at time n (i.e., the beginning of the n-th power update period), the current local mean measurements $y_g(n)$ and $y_I(n)$ are incorporated to obtain the *updated* (*a posteriori*) estimates $\hat{x}_g(n)^+$ and $\hat{x}_I(n)^+$. The two-dimensional filter gain vectors L_g and L_I are obtained as:

$$\begin{aligned} L_g(n) &= P_g(n)^- H_f^T \left(H_f P_g(n)^- H_f^T + V_g\right)^{-1}, & (8.19) \\ L_I(n) &= P_I(n)^- H_f^T \left(H_f P_I(n)^- H_f^T + V_I\right)^{-1}, & (8.20) \end{aligned}$$

where V_g and V_I are the measurement noise covariances and $P_g(n)^-$ and $P_I(n)^-$ are the *propagated* estimation error covariance matrices. Note that we only have scalar measurements and no matrix inversion is involved. At time n, the covariance matrices are updated as:

$$\begin{aligned} P_g(n)^+ &= P_g(n)^- - L_g(n) H_f P_g(n)^-, & (8.21) \\ P_I(n)^+ &= P_I(n)^- - L_I(n) H_f P_I(n)^-. & (8.22) \end{aligned}$$

Now the one-step predicted estimates for the channel gain and the interference level are obtained by propagating the estimates to the next power update period:

$$\hat{x}_g(n+1)^- = A_f \hat{x}_g(n)^+, \tag{8.23}$$

$$\hat{x}_I(n+1)^- = A_f \hat{x}_I(n)^+, \tag{8.24}$$

and the covariance matrices are propagated as:

$$P_g(n+1)^- = A_f P_g(n)^+ A_f^T + W_g, \tag{8.25}$$

$$P_I(n+1)^- = A_f P_I(n)^+ A_f^T + W_I, \tag{8.26}$$

where W_g and W_I are two-dimensional diagonal covariance matrices for the driving noise sequences in (8.11) and (8.12), respectively.

Incorporating the one-step predicted estimates in the integrator algorithm (7.43), the updated power level for the duration of the n-th power update period can be obtained as:

$$\bar{p}(n) = \bar{p}(n-1) + \left(\bar{\gamma} - \hat{\bar{r}}(n+1)^-\right), \tag{8.27}$$

where:

$$\begin{aligned} \hat{\bar{r}}(n+1)^- &= \bar{p}(n-1) + \hat{x}_{g1}(n+1)^- - \hat{x}_{I1}(n+1)^- \\ &= \bar{p}(n-1) + \hat{\bar{g}}(n+1)^- - \hat{\bar{I}}(n+1)^-. \end{aligned} \tag{8.28}$$

As explained in Section 9.2, when a new call arrives in the network and is assigned to a base station (after an initial call set-up time), it is checked whether the user can achieve the required SIR threshold for the new calls on the idle channel which currently has the minimum interference level. If so, the user is admitted to the network and its Kalman filter estimates are initialized as $\hat{x}_{g1}(0)^- = \hat{x}_{g2}(0)^- = \bar{g}(0)$ and $\hat{x}_{I1}(0)^- = \hat{x}_{I2}(0)^- = \bar{I}(0)$, where $\bar{g}(0)$ and $\bar{I}(0)$ are the local mean channel gain and interference values available at the time that the user is admitted. Also the error covariance matrices are initialized as:

$$P_g(0)^- = P_I(0)^- = \begin{bmatrix} \sigma_s^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix}, \tag{8.29}$$

where σ_s is the shadow fading standard deviation (set to 8 dB in our simulations). Moreover, when a base station hand-off or a successful channel reassignment occurs, the error covariance matrices are reset to their initial values and the filter estimates are reset to the corresponding local mean values on the new channel.

We now select the values for the filter parameters. As mentioned before, we implement the same filters for all the users in the network. We pick the model parameter a according to (8.3) and by considering the maximum user velocities that we expect in our mobile environment. This makes the filter assume the least correlation among the local mean values in two consecutive power update periods and therefore rely more on the measurements. As we shall explain in our simulation details in the next chapter, we assume the power levels to be updated every 100 msec. Also we consider the shadowing correlation distance to be about 40m and the maximum user velocity to be 80 km/hr. Using (8.3), we then pick $a = 0.95$. Using this value for a and $\sigma_s = 8$ dB and (8.4), we get $\sigma_{w_g}^2 = \sigma_{w_I}^2 = 1.56$. We choose to set $\sigma_{w_g}^2 = \sigma_{w_I}^2 = 2.0$ in the filter, again to deal with uncertainties in the models. The variances for the fictitious driving noise sequences w_{g0} and w_{I0} are also set to 2.0 dB^2. Also the standard deviations for the local mean measurement errors are both set to 3.0 dB, i.e., $V_g = V_I = 9.0$. These values are summarized in Table 8.1. It should be mentioned that one may try to adaptively obtain estimates for these variances from the available measurements or by looking at the filter residuals [69]. However, due to the level of uncertainty that we expect in the filter models, we prefer the simpler approach of looking at these variances as design parameters that should be tuned to get a desirable

a	V_g, V_I (dB^2)	$W_g(1,1)$, $W_I(1,1)$ (dB^2)	$W_g(2,2)$, $W_I(2,2)$ (dB^2)
0.95	9.0	2.0	2.0

Table 8.1: Kalman Filter Model Parameters

performance. We will see that by using the above values for the parameters, some global stability results are obtained while computer simulations show noticeable improvements in the overall system performance.

Finally, we observe that the error covariance matrices and the filter gains are independent of the actual measurements. This can be seen from the filter equations (8.19)-(8.26). Therefore, the filter gains L_g and L_I can indeed be calculated and saved *a priori*. Since the same filters are used for all the mobiles in the network, most filter calculations should thus be repeated only every once in a while, when updated estimates for some of the environment-related parameters (shadowing correlation distance, etc.) become available. This can result in a considerable reduction in the filter processing time.

Also note that when the filter reaches the steady-state on a specific channel, the steady-state filter gain vectors are equal to:

$$L_g = L_I = PH_f^T \left(H_f P H_f^T + V\right)^{-1}, \tag{8.30}$$

where $V_g = V_I = V$ and P is the positive-definite solution to the following *discrete Riccati equation*:

$$P = A_f P A_f^T - A_f P H_f^T (H_f P H^T + V)^{-1} H_f P A_f^T + W, \tag{8.31}$$

where $W_g = W_I = W$. Using the values given in Table 8.1, we get:

$$L_g = L_I = L = \left[\begin{array}{cc} 0.37990 & 0.37121 \end{array}\right]^T. \tag{8.32}$$

8.4 Global Stability

When the Kalman filters are employed, the block diagram for a single loop can be depicted as in Figure 8.1. We now show that, in the steady-state, the Kalman filters and therefore the local loops are stable[1]. Moreover, the sufficient conditions for global stability are satisfied.

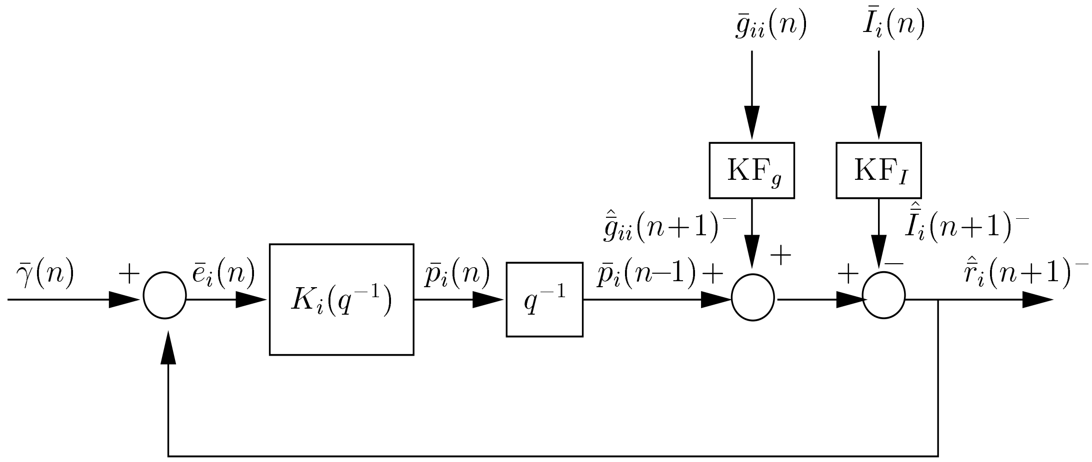


Figure 8.1: A local power control loop with Kalman filters

Given the filter gains in (8.32), it is straightforward to obtain the steady-state transfer functions for the Kalman filters:

$$\frac{\hat{\bar{g}}(n+1)^-}{\bar{g}(n)} = \frac{\hat{\bar{I}}(n+1)^-}{\bar{I}(n)} = \frac{q\,(0.37947q - 0.36091)}{q^2 - 1.57053q + 0.58909} \tag{8.33}$$

The poles of the Kalman filters (i.e., the poles of the above transfer function or equivalently the eigenvalues of $A_f - A_f L H_f$) are located inside the unit circle at:

$$s_{f1} = 0.61928, \qquad s_{f2} = 0.95125 \tag{8.34}$$

It is now clear that all the local loops are stable, i.e., the poles for all the closed-loop transfer functions, associated with a single loop, are inside the unit circle.

[1] Under the technical conditions of stabilizability and detectability, the steady-state Kalman filters are always known to be stable [69]

Processing and propagation delays (i.e., extra delay blocks in the feedback path) could result in instability of the local loops and therefore instability of the whole network. However, even though some delay compensation schemes have been proposed in [31], information-feedback power control algorithms, as mentioned before, usually run on slower power update rates and processing and propagation delays are usually much less than a single power update period.

As we mentioned, stability of the local loops is necessary but not sufficient for global stability of the network. However, the network will indeed be globally stable in the $\ell_\infty - induced$ norm sense, if the transfer function from the interference $\bar{I}(n)$ to the power $\bar{p}(n-1)$, satisfies the norm condition (7.68).

Using (8.33) and from Figure 8.1, it is straightforward to obtain:

$$G(q) = \frac{\bar{p}(n-1)}{\bar{I}(n)} = \frac{0.37947q - 0.36091}{q^2 - 1.57053q + 0.58909}, \tag{8.35}$$

and hence we get:

$$||G(q)||_{\ell_2 - induced} \simeq ||G(q)||_{\ell_\infty - induced} = 1.0. \tag{8.36}$$

Therefore $G(q)$ satisfies both (7.68) and (7.70). From (7.68), we conclude that, as long as the network is in its feasible region, the deviations of the power levels of all the users in the network from their corresponding optimal values will always remain bounded. Moreover, from (7.70), we conclude that if the power levels only slightly deviate from their optimal values, while the channel gains remain constant, they will asymptotically converge back to their optimal values. This proves the global stability of the network, on every channel, both in ℓ_∞ sense and in ℓ_2 sense (with a linearized interference function), when the Kalman filters are at their steady-state.

8.5 Summary

In this chapter, we introduced a novel predictive power control algorithm. We first proposed simple models for the slow variations in the channel gains and the interference levels, and then showed how simple Kalman filters can be designed to provide one-step predicted values for the channel gains and the interference levels in the network. A Kalman filter is associated with the power control loop for every user, while the parameters of the filter are selected to be the same for all co-channel users. The predicted values from the Kalman filters are then incorporated in a distributed integrator power control algorithm. Finally, we showed that the global stability of the network on a single channel is preserved when the Kalman filters are introduced in the power control loops.

In the next chapter, we will provide the simulation results which show the improvement in performance when our predictive power control algorithm is integrated with a minimum interference dynamic channel assignment scheme.

CHAPTER 9

Integrated Predictive Dynamic Channel and Power Allocation

9.1 Introduction

As we have emphasized, with the steadily growing need for capacity in mobile radio systems, optimal allocation of resources in non-uniform and non-stationary environments has become a great challenge. The fundamental objective is to accommodate as many users as possible, subject to complexity and Quality of Service requirements, on a limited available bandwidth by controlling undesired interactions among the users. One major interaction is the co-channel interference that every user generates for all other users, which are sharing the same channel. Various techniques have been developed to mitigate the effects of co-channel interference. Some of these techniques, such as sectorization and beamforming using smart antenna arrays, try to suppress interference, while others such as channel assignment techniques try to avoid strong interferers.

As we explored in the last two chapters, another technique is to adaptively control the power levels of all the users in the network. Power control can be very effective, especially when the network is *interference-limited*, i.e., the number of co-channel users is mostly limited by the amount of interference that they are causing for each other.

At the same time, Dynamic Channel Assignment (DCA) schemes are proposed as another approach to mitigate co-channel interference and increase capacity. While DCA schemes add to complexity of the system, they can achieve higher levels of capacity by adaptively redirecting the traffic to the channels with better conditions.

It is believed that an aggressive DCA scheme can make an FDMA/TDMA system an interference-limited system. This fact has initiated research on integrated distributed Dynamic Channel and Power Allocation (DCPA) schemes [14, 26, 53, 78]. In [14] a pilot based minimum interference DCA scheme is integrated with a fast fixed-step power control algorithm, while fast fading and user mobility effects are neglected. In [53] three different types of minimum interference DCA algorithms are integrated with a slow integrator power control algorithm. Pedestrian mobility along with a low power update rate are considered, and it is again assumed that the fast fading effects are averaged out. In [78] a simulation study has been performed to investigate the joint effects of some simple SIR-based and signal-level-based power control algorithms along with a minimum interference channel reassignment scheme. Fast fading effects are again neglected and low power update rates (1/(0.48 sec) corresponding to GSM measurement intervals) are assumed.

We note that all the above results only consider simple power control algorithms. Moreover, except for [53][78], other results neglect such effects as dynamics of user arrival or departures, user mobility, and base station hand-offs. Our main objective in this chapter is to investigate the performance of our predictive power control algorithm when it is integrated with a minimum interference DCA scheme. We set up a system-level simulation platform, similar to the ones presented in [14][53], to compare our predictive DCPA scheme with the one that uses a simple integrator power control algorithm with no prediction. Dynamics of user arrivals and departures, user mobility and base station hand-offs are all considered in this study. Slowly varying flat Rayleigh fading effects are also considered in our simulations.

The organization of this chapter is as follows. In Section 9.2 we review Dynamic Channel assignment schemes. Specifically, we review the minimum interference scheme and mention how it results in minimum transmit powers for the new incoming users. Then in Section 9.3 we explain all the details of our simulation model. These include the propagation model, models for user arrivals and mobility, the admission control scheme, channel reassignments and base station hand-offs, and the details of our DCPA scheme. In Section 9.4 we discuss the simulation results and compare the performance of our integrated predictive DCPA algorithm with the corresponding algorithm which uses no prediction. We show that, for a range of traffic loads, the number of blocked calls and the number of dropped calls are decreased under our predictive DCPA scheme. Moreover, on average, fewer channel reassignments are required for every call, implying a more stable network. We will provide concluding remarks in the final section.

9.2 Dynamic Channel Assignment (DCA)

Fixed Channel Assignment (FCA) is the traditional approach in FDMA/TDMA systems. In this approach, by considering the worst case scenarios for the traffic load in the network, a pattern for a reuse cluster is selected and every base station in the cluster is then assumed to permanently have access only to a subset of the available channels. This is a simple conservative approach, which does not take into account the current spatial and temporal traffic distributions in the network. On the other hand, in Dynamic Channel Assignment (DCA) schemes, all base stations have access to all the channels and dynamically assign the channels to the users based on the current traffic conditions. While DCA schemes are clearly more complicated, they usually result in higher capacity. Various centralized and decentralized DCA schemes have been proposed in the literature [17, 28, 47, 72].

We adopt the distributed minimum interference DCA scheme [28]. In this scheme, the new users will be assigned to the idle channels with minimum local mean interference, in the order they arrive. It was shown in Section 7.4 that when a new user is admitted to a power-controlled network, the optimal power level for the new user can be written as:

$$p_n^* = \frac{I_{n0}}{g_{nn}} \frac{\gamma_n}{1 - \frac{\gamma_n}{\gamma_{max}}}, \tag{9.1}$$

where γ_n is the SIR threshold that the new user wants to achieve, γ_{max} is the maximum achievable SIR for the new user and I_{n0} is the local mean interference plus noise level at the intended base station of the new user before it is admitted to the network. It is now clear that the minimum interference DCA scheme does indeed result in the minimum transmit power for the new user.

Channel reassignment schemes are required to adaptively redirect the traffic to the channels with better conditions. In fact, whenever the local mean SIR for a user drops below a given threshold, while the user is transmitting at its maximum power level, a channel reassignment attempt is triggered and, if possible, the user is reassigned to the idle channel, which currently has the minimum local mean interference. Note that this is a distributed scheme, since every base station only needs to keep track of the local mean interference values on all of its own idle channels and no communication is necessary among the base stations. In other words, no global reassignment of all the channels in the network is considered when a new user arrives. Therefore, the channel assignments are not globally optimal. However, any kind of global optimality in the channel assignments can only be achieved through centralized algorithms, which are usually impractical due to the excessive requirements for processing and also communication among the base stations.

Note that integrating a minimum interference DCA scheme with a power control algorithm can also be justified by considering the fact that it results in relatively weak coupling among the users in the network. In other words, the channels are selected such that the channel gain matrix becomes as diagonally dominant as possible with a distributed channel assignment scheme. As we saw earlier, this in turn implies that higher SIR thresholds can be supported in the network.

Another issue is call management and admission control. As we discussed in Section 7.4, a network should be feasible for every user to be able to achieve its desired SIR threshold. If no admission control is employed, a new user could potentially force the network out if its feasibility region and hence result in the more unfavorable event of dropping active calls. In [7] an admission algorithm was presented for a power controlled system, where the new users would increase their powers only in small steps. It was shown how this scheme could protect the quality of active links when new users arrive. Channel probing techniques were proposed in [6, 33, 80] where a new user would try to estimate the maximum SIR level that it can achieve by disturbing the network as little as possible. The user will then be admitted only if its maximum achievable SIR is above its desired threshold. Also a channel partitioning scheme was presented in [34] where a combination of dynamically allocated and fixed assigned channels are incorporated to develop a rapid distributed access algorithm.

We adopt the simpler threshold-based implicit admission control scheme, presented in [53]. In this scheme, a new user with a desired SIR threshold γ_d will be admitted only if there exists an idle channel, on which it can achieve an SIR threshold γ_{new}, which is higher than γ_d by a given protection margin. The value of the protection margin for new users should be selected based on the trade-off between blocking new calls and dropping active calls.

Moreover, a channel reassignment attempt will be triggered for a user if, while transmitting at the maximum power, its local mean SIR drops below a threshold γ_{min}, which is lower than γ_d by another given margin. This margin is required to avoid immediate channel reassignments when new users are admitted or when the local mean SIR's for the users temporarily drop due to user motions and fading. The value of this margin should be selected according to the trade-off between quality of service and the average number of channel reassignments per call. Note that for channel reassignment, it is checked whether the user can achieve γ_d on the idle channel which currently has the minimum interference. Since $\gamma_d < \gamma_{new}$, this scheme clearly favors the active users, that are being reassigned, to the new incoming users. If a channel reassignment fails, the user stays on its old channel and the reassignment attempt is repeated every reassignment period (as long as $r < \gamma_{min}$ and $p = p_{max}$) until the user is either successfully reassigned or dropped from the network. Finally, a user will be dropped from the network if its local mean SIR drops and stays below a threshold $\gamma_{drop}(< \gamma_{min})$ for a given duration of time.

9.3 Simulation Model

While the theoretical analysis in Chapter 8 helps in justifying the use of Kalman filters in power control algorithms to deal better with the variations in the channel gains and the interference levels, and also the errors in the local mean measurements, a simulation study is essential to analyze the overall performance when such a predictive power control algorithm is integrated with a DCA scheme in a relatively realistic mobile radio environment. We therefore decided to set up a system-level simulation environment, similar to the ones presented in [14][53] but on a smaller scale, in order to analyze the overall performance of the network, when our predictive power control algorithm is integrated with a distributed minimum interference DCA scheme. User arrivals and departures and user mobility are all considered in this study. In this section, we explain the details of our simulation platform, and in the next section we analyze the results.

9.3.1 Basic Assumptions

We consider a cellular system where the area under coverage is divided into cells and each cell has its own base station. All users communicate with their assigned base stations through a single hop. This is in contrast to *ad hoc* wireless networks where there is no fixed infrastructure and multi-hop communication is prevalent.

We focus on a Frequency/Time Division Multiple Access (FDMA/TDMA) system, where each channel is characterized by a pair (m, n) where m denotes the carrier frequency and n denotes the time slot. We consider two carrier frequencies and eight slots per carrier. We do not consider any blind slots in the system, that is, we assume that any slot in a frame can be used as a traffic channel. Therefore, we have 16 traffic channels in the system. Blind slots can be avoided either by appropriate structuring of the control channel or by assuming that a call activity detection scheme is employed such that the users can temporarily discontinue their transmission in their active slots. Modifying the frame structure and considering some slots as the blind slots should not have major effects on our performance comparisons.

We only consider the co-channel interference among the users, i.e., no adjacent channel interference is assumed. Specifically, we assume a system-wide synchronization to the slot level so that each user will experience interference only from the users which are sharing exactly the same slot on the same carrier frequency. This assumption implies that large enough guard times per slot are assumed.

For simplicity and to avoid complexities like Time Division or Frequency Division Duplexing, and the uplink-downlink interference imbalance, we choose to consider only the uplink channel, i.e., the channel from mobile stations to base stations. We should note, however, that almost all the results and discussions could similarly be stated for the downlink channel.

We assume a fixed-power control (pilot) channel on the downlink. As we shall see, this channel facilitates Dynamic Channel Assignment (DCA) and can be used by the mobiles for initial base station assignments and base station hand-offs.

We abstract the system architecture, as far as modulation, coding, etc. are concerned, and consider Signal to Interference plus Noise Ratio (SIR) as the only measure for Quality of Service (QoS) in the network. This is a common practice, even though Bit Error Rate or Frame Error Rate are usually considered as the ultimate performance measures. The reason is that, in general, higher SIR will result in better bit error rate performance and considering SIR as the measure for quality of service provides us with a more convenient platform for power control design.

While we do not restrict ourselves to any specific standard, we have tried to stay close to the *Global System for Mobile Communications* (GSM) standard.

The system is simulated on the frame level (4.0 msec, assuming 8 slots of 0.5 msec in each frame). It is assumed that the signal and interference power measurements for every user are available in every frame at the end of the user's corresponding slot. Various events might then happen every multiple number of frames.

9.3.2 Cell Layout and Base Station Antennas

A 3x3 square grid of cells is assumed. The base stations are located on the cell centers and are separated by 800m. To avoid edge effects, a ring simulation structure is assumed, i.e., the statistics are only gathered from the central cell. This is somewhat simpler than a toroidal simulation structure and is shown to provide more optimistic but comparable results [54]. The other reason for our results to be somewhat optimistic is that only nine cells are simulated, and therefore lower interference levels are generated. However, our simulation results clearly serve our purpose of comparing our predictive DCPA scheme with the one that uses no prediction.

Omni-directional antennas with two branch selection diversity are assumed for the base stations. To simulate the selection diversity, two uncorrelated Rayleigh fading components are generated for every user and the greater one is picked.

9.3.3 Propagation Model

The channel gain for every link is normalized with respect to the base station and mobile antenna gains and is characterized by three components: distance loss, slow or shadow fading and fast fading.

- Distance Loss

  The distance loss is assumed to be inversely proportional to d^α, where α is called the propagation exponent and is set to 4.0.

- Slow or Shadow Fading

A log-normal shadowing pattern is generated *a priori*. Therefore the shadowing values only depend on the user's location. The shadowing correlation distance X_s is assumed to be 40m. Gaussian numbers with correlations given by (8.4) (with $a = 1/e$) are generated every correlation distance on a square grid. The shadowing for every user is then obtained by a normalized bilinear interpolation of the four closest points of the shadowing grid. This interpolation preserves the log-normal distribution and the variance of the shadowing.

- Fast Fading

  A slowly varying flat Rayleigh fading is assumed. This implies that no line-of-sight exists and the delay spread is small compared to the symbol duration or inverse channel bandwidth and thus only a single path with a Rayleigh distributed amplitude (and hence exponentially distributed power) can be distinguished. In fact, the Rayleigh fading component is assumed to be constant for the whole duration of a single slot (0.5 msec). Time correlation for Rayleigh fading is often represented using the Jake's model [43], where it is expressed in terms of a zero order Bessel function of the first kind, which results in a non-rational spectrum. We use a first-order approximation by passing a white complex Gaussian noise through a first order filter and obtaining the squared magnitude of the output Gaussian process. The time constant of the filter, for every user, is obtained by setting its 3 dB cut-off frequency equal to $f_m/4$ where $f_m = v/\lambda$ is the maximum Doppler frequency for the user [72].

Note that even though the instantaneous channel gains for every user are different on different channels, we consider the local mean values (that are used for base station and channel assignment or reassignments) to be the same for all channels (including the control channel) in a base station.

9.3.4 Call Arrivals and Holding Times

New calls are generated based on a Poisson process with a given arrival rate λ_a. Each call is assigned an exponentially distributed holding time with a given average value T_h. The average *Erlang* load per cell is then obtained as:

$$E_c \triangleq \frac{\lambda_a T_h}{N_c}, \tag{9.2}$$

where $N_c = 9$ is the total number of cells. The Erlang load per cell effectively determines the average number of users that could be active in every cell at any instant of time. We have considered various combinations of values for λ_a and T_h to simulate the network under different traffic load conditions.

9.3.5 Mobility Model

The new users are uniformly distributed in the area. The mobility of the user i is modeled with a constant but random speed v_i and the angle θ_i between the velocity vector and the horizontal axis ($-\pi \leq \theta_i < \pi$). The speed for every new user is selected randomly from a triangular distribution in the range 0-80 km/h (i.e., with the mean 40 km/h). This is preferred over a uniform distribution, as it results in a smaller variance for the velocity distribution among different users. The initial direction θ is uniformly picked. Then every 10 sec, a new direction is selected from a triangular distribution with the old direction as its mean. This is again preferred over a uniform distribution or a two dimensional random walk, since it makes small angle turns more probable that large ones. As mentioned before, we use a ring simulation structure. Therefore, when a user wants to cross a border of the area, it will bounce back. The motion trajectory for a sample

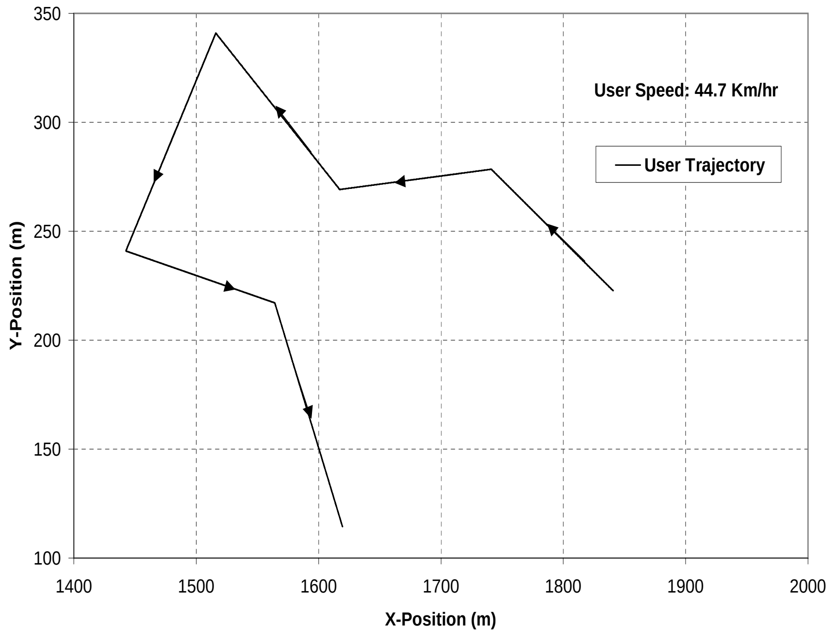


Figure 9.1: A Sample User Motion Trajectory

user is shown in Figure 9.1.

9.3.6 Admission Control

As explained in Section 9.2, a simple threshold-based admission control scheme is employed. The desired SIR threshold for all users in the network is set to $\bar{\gamma}_d = 12$ dB, while the minimum tolerable SIR is considered to be $\bar{\gamma}_{min} = 10$ dB. Both margins for new user admissions and user droppings are set to 2 dB. Therefore new users will be admitted only if they can achieve $\bar{\gamma}_{new} = 14$ dB on the idle channel with the minimum local mean interference. Moreover, a user will be dropped from the network if its SIR drops below $\bar{\gamma}_{drop} = 8$ dB and stays below for four consecutive seconds. Note that these margins should have been expressed as percentages of $\bar{\gamma}_d$ and $\bar{\gamma}_{min}$ for every user, if the users were to have different quality of service requirements and thus different SIR thresholds.

9.3.7 Base Station and Channel Assignments and Reassignments

When a new user arrives into the network, it first starts scanning the downlink control channel from all neighboring base stations and measures all the local mean channel gains. It is assumed that this process take about 0.8 sec (200 frames), which is called the initial call set-up time. The new user then sends its request for a channel to the base station which has the strongest signal. If this base station does not have any idle channels, the user will try the second best base station. This procedure is called *Direct Retry* and will be repeated for a given number of base stations (set to 3 in our simulations) before the user is blocked. When there are idle channels available, the base station checks whether the user can achieve γ_{new} on the idle channel with the minimum local mean interference. If so, the user will be admitted and will be assigned to the idle channel with the minimum interference. Otherwise, the user will be blocked.

We should note that no *macro diversity* is considered, i.e., any user will only communicate with a single base station at any instant of time. Moreover, base station assignment is considered to be separate from power control, i.e, the power levels are obtained assuming that the users are already assigned to their corresponding base stations. Joint base station assignment and power control has already been proposed in the literature [88].

A minimum interference DCA scheme, as explained in Section 9.2, is employed. The local mean channel gain and interference values for possible channel reassignments are obtained by simple averaging of the available measurements over 50 consecutive frames for every user, that is, the channel reassignments for every active user can happen every 200 msec.

Finally, a base station hand-off attempt will be triggered if the local mean channel gain from a neighboring base station exceeds the corresponding value from the current base station by a selected hand-off margin of 4 dB. If the hand-off attempt fails, the user will stay with its current base station. Note that the users are assumed to be continuously monitoring the downlink control channels of all neighboring base stations.

9.3.8 Power Control

The simple integrator algorithm in (7.43) and (7.44) and the predictive algorithm in (8.27) and (8.28) are simulated and compared. Note that while the propagation simulation models are tailored to the individual users, according to their different trajectories and speeds, the same Kalman filter models and parameters are employed for all the users in the network. After a new user i is admitted, it sets its initial power at:

$$p_i(0) = \frac{\gamma_d I_i(0)^-}{g_{ii}(0)^-}, \tag{9.3}$$

where $I_i(0)^-$ and $g_{ii}(0)^-$, respectively, denote the local mean channel gain and interference plus noise level, which are available at the time of user admission. Note that this is somehow an optimistic choice, since a new user sets its initial power as though other users will not increase their transmit powers.

The power update rate is assumed to be the same for all users and is set to 100 msec, that is, every user updates its power level every 25 frames according to (7.43) or (8.27). Note however that a maximum transmit power constraint at 30 dBm is imposed on all users in the network, while the receiver noise floor is set to -120 dBm.

Since the users arrive at arbitrary instants of time according to a Poisson arrival process, the power updates are, in fact, performed *asynchronously*, even though all the users have the same power update rates. While most results in power control assume synchronous power updates among the users, asynchronous power control algorithms have been addressed in the literature [87]. To have synchronous power updates, one could simply force the users to arrive at instants of time, which are multiples of the common power update period.

In the next section we present and analyze our simulation results and show how the predictive DCPA scheme can improve the overall performance of the network.

9.4 Performance Analysis

Our objective in this section is to use our simulation results in order to compare our predictive DCPA scheme with the one that uses a simple integrator algorithm with no prediction, in a non-stationary mobile environment. The two algorithms are simulated in exactly the same environment and under exactly the same assumptions and parameter values.

We base our comparisons on four aspects of performance. First, we compare call blocking and call dropping probabilities. Then we look at those measures that show how good the users in the network can achieve their desired SIR thresholds. The average number of channel reassignments per call is then compared under the two algorithms. Finally, we look at the transmit power distributions of the users in the network. While each of these performance measures gives us insight on specific aspects of the overall network performance under the two DCPA algorithms, they are all related and they all contribute to the network capacity, which can be seen as the ultimate measure for any multi-user communication system.

For any given traffic load, we run the simulations multiple times with different random generator seeds and every run continues until enough number of calls are dropped. The statistics are then gathered from the central cell.

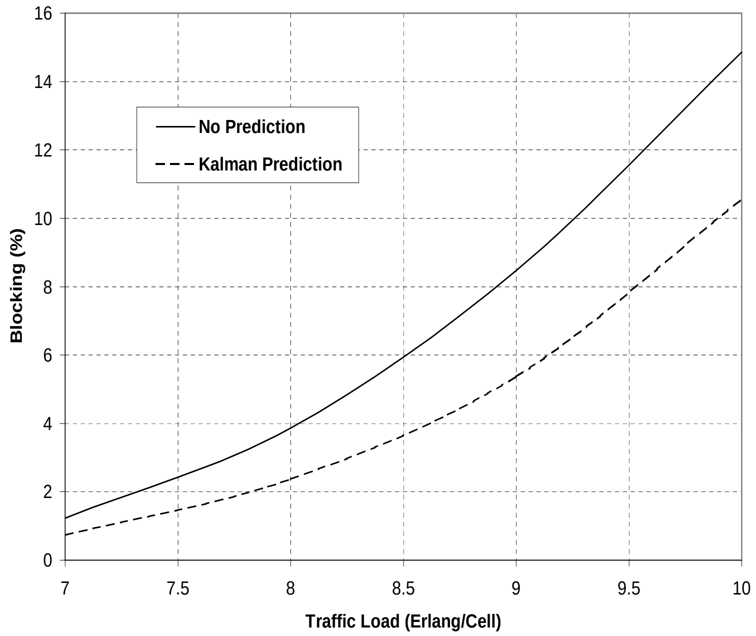


Figure 9.2: Call Blocking Response

Figures 9.2 and 9.3 show the call blocking and the call dropping responses of the network under the two DCPA schemes. It can be seen that at 7.0 Erlang/Cell, the predictive DCPA scheme achieves about 0.5% lower blocking rate and about 0.03% lower dropping rate. Moreover the improvement in performance increases as the traffic load increases. Remember that there is always a trade-off between blocking new calls and dropping active calls.

The local mean SIR for a sample user is shown in Figure 9.4. It can be seen that the predictive algorithm results in a smoother behavior for the local mean SIR.

However, a more appropriate approach to compare the target SIR tracking performance for the two schemes is to look at the SIR error standard deviation, which shows how the local mean SIR values for the users are spread around the target SIR value $\gamma_d = 12$ dB. We obtain an estimate for the SIR error standard deviation and also estimates for the SIR cumulative distribution functions by

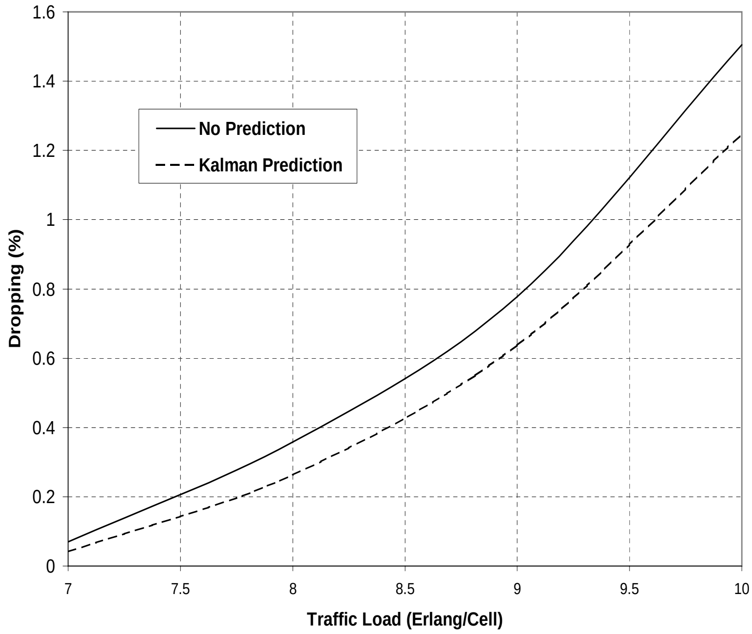


Figure 9.3: Call Dropping Response

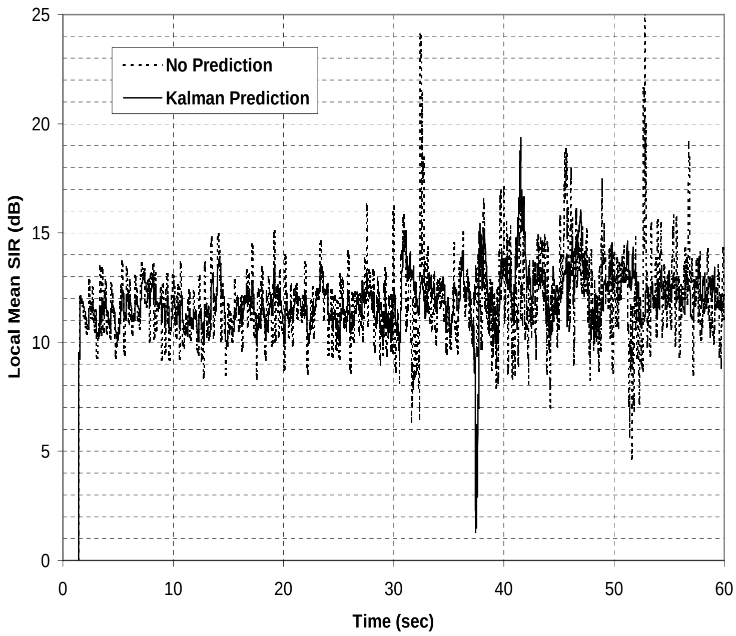


Figure 9.4: The local mean SIR for a sample user moving at 44.7 km/hr

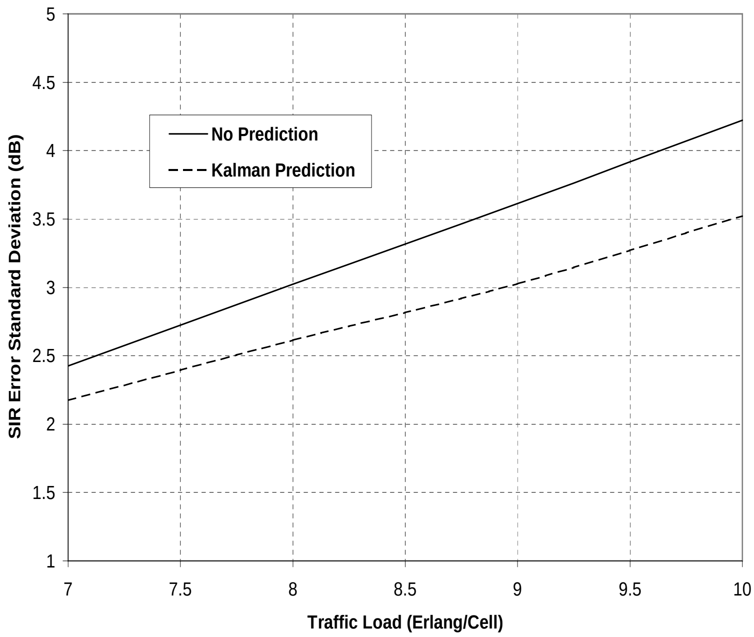


Figure 9.5: Standard Deviation for the Error in the Local Mean SIR

looking at the local mean SIR values of all the users in the network at various random instants of time (after enough call attempts have been made and the network has reached some kind of steady state) during every run of the simulation. Figure 9.5 shows the standard deviation for the error in the local mean SIR for a range of traffic loads. It can be seen that the predictive scheme decreases the SIR error standard deviation by about 0.3 dB at 7.0 Erlang/Cell, while the improvement is about 0.7 dB at 10.0 Erlang/Cell.

Figures 9.6 and 9.7 show the cumulative distribution for the local mean SIR values in the network under two different traffic loads. It is shown that the predictive DCPA scheme results in the local mean SIR values, which are less spread around the target SIR. The improvement is again more noticeable in higher traffic loads.

In fact, Figures 9.8 and 9.9 show how the local mean SIR cumulative distri-

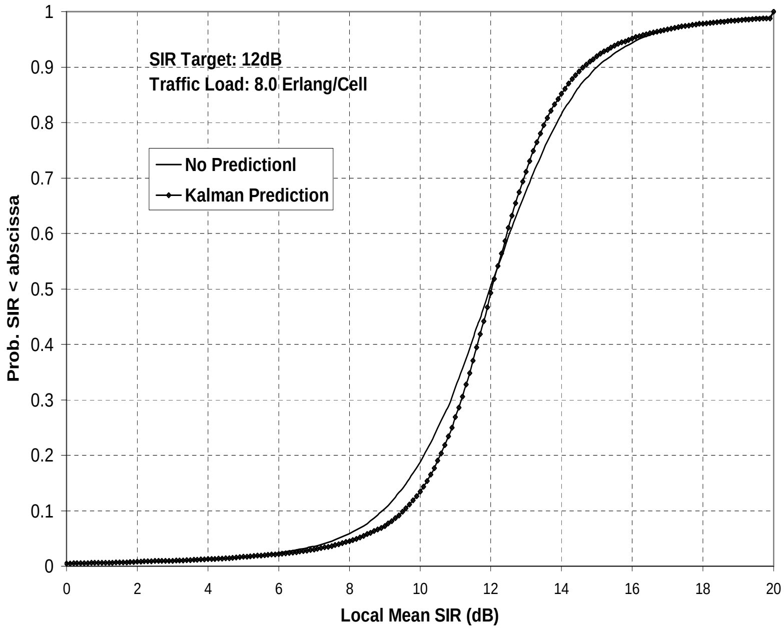


Figure 9.6: Cumulative Distributions for Local Mean SIR at 8.0 Erlang/Cell

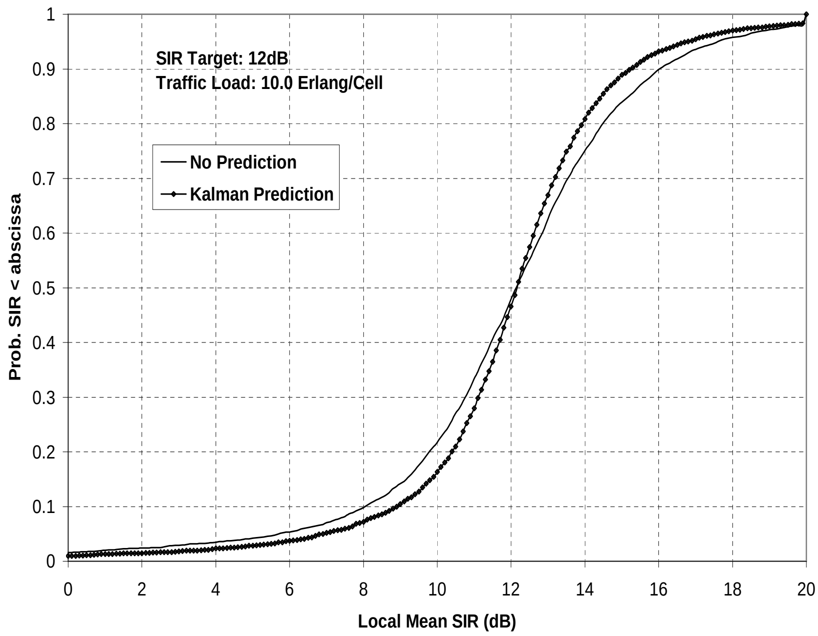


Figure 9.7: Cumulative Distributions for Local Mean SIR at 10.0 Erlang/Cell

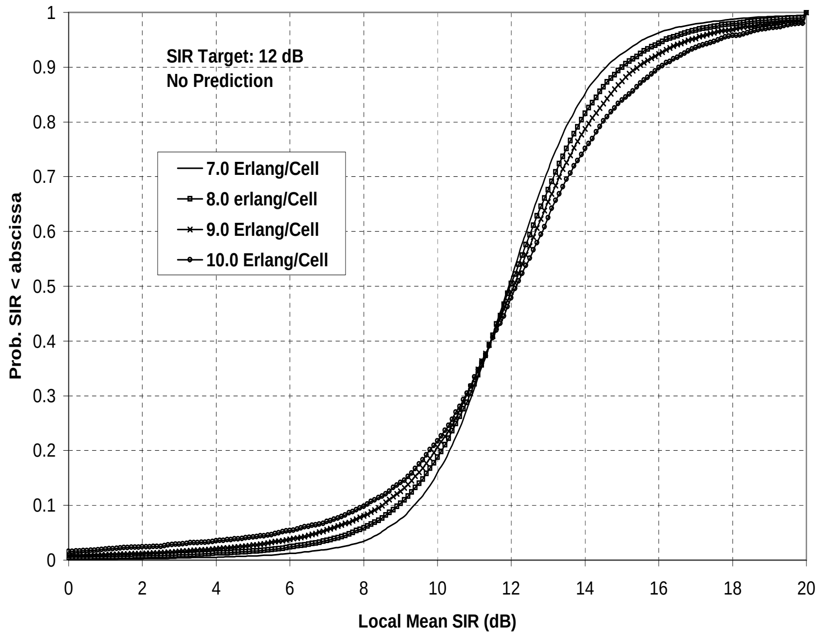


Figure 9.8: Traffic Load Effect on Local Mean SIR Cumulative Distribution (No Prediction)

bution functions change with the traffic load under both schemes.

As mentioned before, one measure that indicates the level of stability of the network is the average number of channel reassignments per call. Figure 9.10 shows this number for a range of traffic loads under both DCPA schemes. As one would expect, fewer channel reassignments per call are, in average, required in the predictive DCPA scheme. One reason for this is that, as shown before, the predictive scheme does in fact result in better target SIR tracking and smoother local mean SIR behavior.

Finally, we compare the transmit power distribution of the users in the network under the two DCPA schemes. Figure 9.11 shows an estimate of the cumulative distribution function for the transmit powers of the users in the network at the load of 8.0 Erlang/Cell. It can be seen that the two schemes perform quite similarly, as far as transmit powers are concerned. In fact, both algorithms result

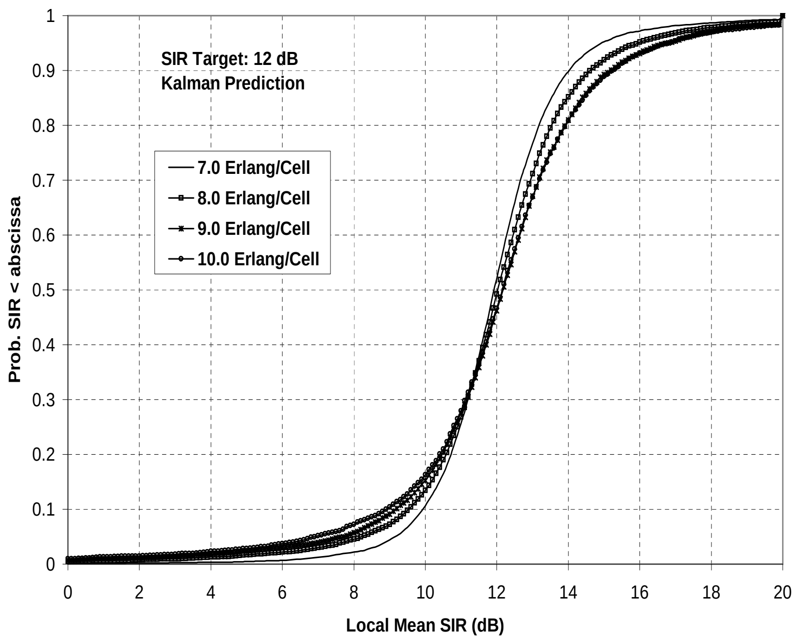


Figure 9.9: Traffic Load Effect on Local Mean SIR Cumulative Distribution (Kalman Prediction)

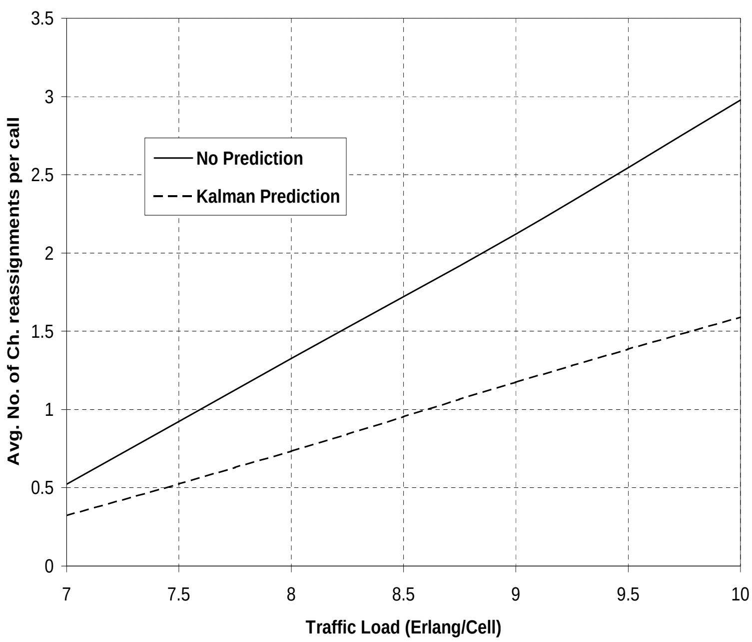


Figure 9.10: Average Number of Channel Reassignments per Call

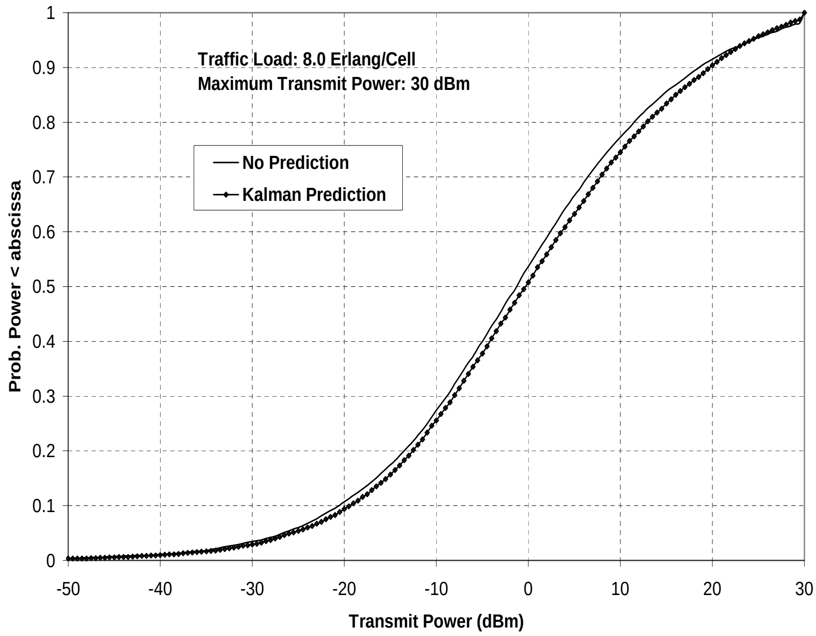


Figure 9.11: Transmit Power Cumulative Distribution at 8.0 Erlang/Cell

in considerable power saving, when compared with a network where all the power levels are fixed at their maximum levels. For example, at a relatively high load of 8.0 Erlang/Cell, about 50% of the users under both DCPA schemes are transmitting at 0 dBm or lower power levels. It should however be mentioned that our predictive DCPA algorithm seems to result in slightly higher power levels in the network. This could presumably be because we admit more calls in the network and thus push closer to capacity. One may also see this as a small cost for better SIR tracking and better call blocking and dropping responses. Figures 9.12 and 9.13 show how the power cumulative distribution functions might change as the traffic load on the network changes.

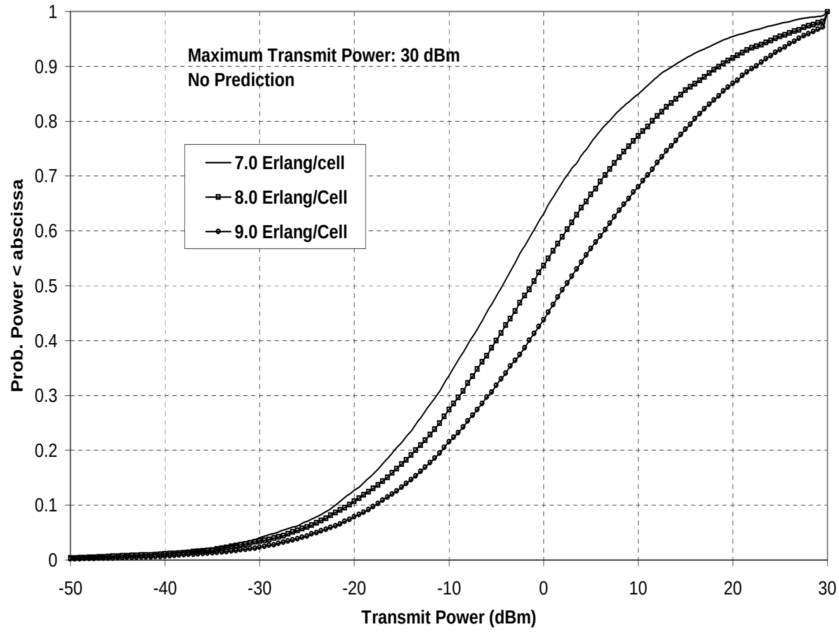


Figure 9.12: Traffic Load Effect on Transmit Power Cumulative Distribution (No Prediction)

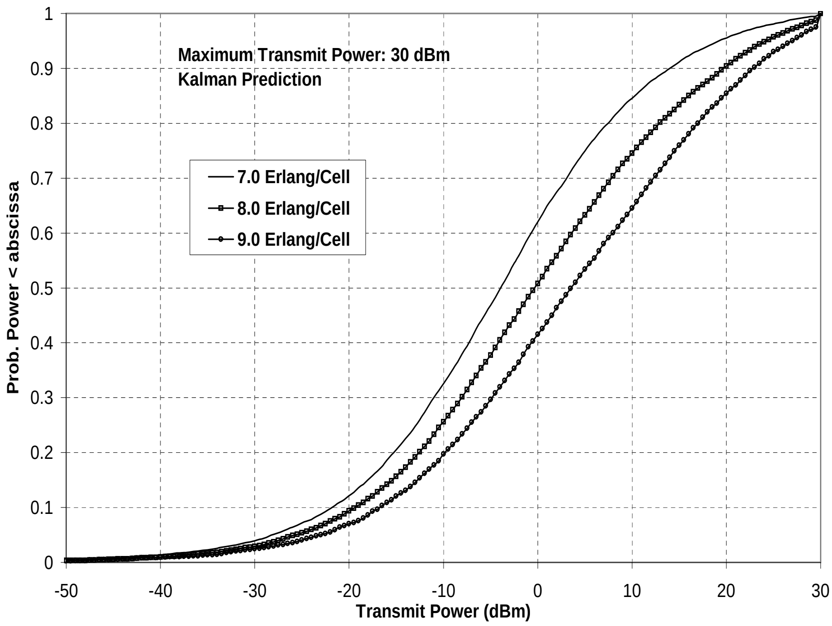


Figure 9.13: Traffic Load Effect on Transmit Power Cumulative Distribution (Kalman Prediction)

9.5 Summary

A predictive Dynamic Channel and Power Allocation scheme was presented in this chapter. In fact, the predictive power control algorithm, which was presented in Chapter 8, was integrated with a Minimum Interference Dynamic Channel Assignment scheme in an FDMA/TDMA mobile radio system. A system-level simulation environment was then developed. User arrival and departures and user mobility along with flat Rayleigh fading effects were all included in the simulations.

It was shown that the predictive DCPA scheme results in better call dropping and blocking responses and also better target SIR tracking performance for the network. Moreover, in average, fewer channel reassignments per call are required under the predictive DCPA scheme. We believe that these improvements are obtained mainly because the predictive algorithm takes into account at least the slow variations of the channel gains. Also by dealing with uncertainties in the measurements, it effectively mitigates the fading induced local mean measurement errors. It was shown however that the predictive DCPA scheme results in slightly higher power levels for the users in the network.

As for future research, one may try to design adaptive algorithms where the parameters of the algorithm and even the power update rates are adaptively adjusted for individual users, according to such information as user velocities, etc. Also the standard integrator algorithm may not be the best power control algorithm. Even though constraints on complexity and computational effort are always present, other simple algorithms may still be designed that could result in better SIR tracking, better allocation of resources and ultimately higher capacity in highly non-uniform and non-stationary environments.

Part V

Concluding Remarks

CHAPTER 10

Conclusions and Future Research

With tremendous growth in computer, communication, control, and network technologies in recent years, various industrial and research institutions have started exploring possibilities for designing large scale systems, which are composed of multiple autonomous local subsystems. Decentralized systems can generally be very efficient and highly robust to faults and uncertainties. At the same time, however, there are many difficulties in designing distributed control and coordination algorithms. Our objective in this research was to address some of these difficulties.

The broad range of applications for decentralized systems has inspired engineers and researchers from many different fields; from economists to control theorists and computer scientists. In this research, we mostly focused on applications in control and communication systems.

We started by reviewing some fundamental results in team theory. Team theory was originated in mathematical economy and was one of the first areas where distributed decision making was addressed. Then we looked at the general formulation of decentralized control problems and elaborated on the relations between a team problem and a decentralized control problem.

This led us to the concept of information patterns. In fact, we tried to address two major problems in designing decentralized control algorithms. The first problem was to investigate how the information structure of a decentralized system could affect the control strategies for the local stations. Remember that every station only has access to its local information, which is obtained either by local measurements or through noisy communication with other stations. The way the information is distributed in the system can greatly affect the control strategies and hence the overall performance for the global system. The second problem was to develop reasonable schemes for evaluating a piece of information in a decentralized system. One would like to know how valuable a piece of information can be based on how it can affect the overall performance. Such schemes for information evaluation could be essential in designing information patterns, i.e., distributing information among the local stations. We mentioned how the entropy approach might provide us with a convenient platform for this purpose.

We then focused on the classical Linear Quadratic Gaussian (LQG) control problems and proposed covariance-based schemes for information evaluation. First, we showed how the optimal performance index can change with the measurement noise covariances and thus how non-critical measurements (i.e., the ones with no effects on the system detectability) can be evaluated simply by looking at their corresponding noise covariances.

For future research, one should try other possible schemes. The ultimate objective is to come up with a unified framework for control-oriented information evaluation, especially for decentralized systems. The major obstacle, however, is that performance evaluation can be very application specific, and this makes it very hard to obtain a unified platform with a reasonable level of abstraction.

In the next step, we looked at a two station decentralized LQG problem. We explored a sub-optimal approach in designing decentralized control laws. Namely, we treated the problem as two separate centralized LQG problems. We then considered various cases where the two stations communicate different pieces of information. We showed that even if the stations communicate all their measurements, our sub-optimal controllers may fail to stabilize the global system. However, closed-loop stability can be achieved if the stations communicate their control values. This problem again shows how the distribution of information in a decentralized system can affect the overall performance.

One direction for possible future research is to consider other control frameworks. For example, one can follow a robust control approach and explore the effects of various pieces of information on robustness of the global system. In other words, one could consider a simple decentralized system and investigate whether communicating different pieces of information between the stations can increase the robustness of the global system to the uncertainties in the models of the local stations.

To further explore the effects of non-classical information patterns, we studied a seemingly simple example, proposed by Witsenhausen in 1968. It is a two stage stochastic problem with linear dynamics, additive and Gaussian uncertainties and a quadratic cost. We showed how the non-classical structure of the information pattern transforms the problem into a non-convex functional optimization problem. We then proposed a reformulation of the problem, where the two station start communicating through a noisy channel. We showed that any uncertainty in the communication among the station can again induce a non-classical information pattern.

We then considered a special case where the communication uncertainty is very small. We proposed an asymptotic approach based on which we obtained a necessary condition for the optimal strategies. We showed that the linear strategies still satisfy the necessary condition.

Many researchers have explored different aspects of Witsenhausen's counter-example. However, the results are still far from convincing. Working more on this example or possibly coming up with similar examples can help in dealing with non-classical information patterns, which seem to be the bottleneck for decentralized optimization problems.

The next major part of our research was focused on a specific application in wireless communication systems. Namely, we studied, in detail, the power control problem for cellular radio networks. Power control has been proposed as an effective scheme for co-channel or multiple access interference mitigation in cellular systems.

This is again a decentralized stochastic problem, where every user acts as a local station and only has access to its own set of noisy measurements. We reviewed SIR balancing and SIR threshold as the two main approaches for power control design and showed how the two approaches can be unified. Furthermore, we showed how the power control problem can be posed as a decentralized regulation problem. We used a robust control framework to obtain a sufficient condition for global stability of the network under a given power control algorithm. This sufficient condition would guarantee that the deviations of the power levels of the users from their corresponding optimal values will always remain bounded. As for future research, one could quantify some bounds on such deviations.

In order to deal with the variations in the channel gains and also the errors in the local mean SIR measurements, we proposed a predictive power control algorithm. We showed how simple white noise driven first order Markov models may be used to model the slow variations in the channel gains and the interference levels. Moreover, we proposed to use such models in order to design simple Kalman filters, which would provide us with the one-step predicted values of the channel gains and the interference powers. Such predicted values were then incorporated in an integrator power control algorithm.

We used the same filters for all the users in the network. As future research, one can explore the possibility of implementing adaptive prediction filters for every individual user in the network. In other words, how any extra information, such as user velocities, can be used in order to adaptively change the model parameters or even the power update rate for every user.

Finally, we set up a detailed system-level simulation platform where various integrated Dynamic Channel and Power Allocation (DCPA) schemes can be simulated and compared in relatively realistic mobile environments. We integrated the minimum interference dynamic channel assignment scheme with our predictive power control algorithm. We showed that, comparing with the corresponding DCPA scheme with no prediction, better SIR tracking and better call dropping and call blocking responses can be obtained. Moreover, on average, fewer channel reassignments per call are required, implying a more stable network.

Dynamics of user arrivals and departures along with user mobility were all considered in our simulations. In fact, our simulation platform can be very helpful for any future research on various DCPA schemes. Namely, many different power control algorithms, including the fixed-step algorithms, along with various dynamic channel assignment schemes can be simulated and compared.

Finally, another possible direction for future research is to look into other similar applications in control of networks and try to elaborate on any extendable results. In fact, the bottom-up analysis approach seems to be a very good alternative approach to tackle some of the fundamental open issues in decentralized stochastic systems. In other words, by focusing on specific applications, one might be able to obtain useful results, which can be taken to higher levels of abstraction and can help in formulating a general structure for decentralized control algorithms.

If studying this thesis makes you appreciate some of the fundamental difficulties that exist in controlling distributed networks, and encourages you to get involved with research on this fascinating and rapidly evolving area, our goal is achieved.

References

[1] Altman, E., and Basar, T., "Multiuser rate-based flow control," *IEEE Trans. on Comm.*, Vol. 46, No. 7, pp. 940-49, 1998.

[2] Andersin, M., and Rosberg, Z., "Time variant power control in cellular networks," in *Proc. 7th IEEE Intl. Symp. on Personal, Indoor and Mobile Radio Communications (PIMRC),* Vol. 1, pp. 193-197, October 1996.

[3] Athans, M., "The matrix maximum principle," *Information and Control*, No. 11, pp. 592-606, 1968.

[4] Baglietto, M. and Parisini, T. and Zoppoli, R., "Nonlinear approximations for the solution of team optimal control problems," *Proceedings of CDC 1997*, Vol. 5, pp. 4592-4594, 1997.

[5] Baglietto, M. and Parisini, T. and Zoppoli, R., "Neural networks for the solution of information distributed optimal control problems," In *Proc. of 6th European Symp. on Artificial Neural Networks, ESANN'98*, pp. 79-84, 1998.

[6] Bambos, N. D., Chen, S. C., and Mitra, D., "Channel probing for distributed access control in wireless communication networks," in *Proc. GLOBECOM'95.*

[7] Bambos, N. D., Chen, S. C., and Pottie, G. J., "Radio link admission algorithms for wireless networks with power control and active link quality protection," Technical Report UCLA-ENG-94-25, Elect. Eng. Dept., Univ. of California, Los Angeles, 1994.

[8] Bansal, R. and Basar, T., "Solutions to a class of Linear Quadratic Gaussian stochastic team problems with non-classical information", *Systems and Control Letters*, Vol. 9, pp. 125-130, 1987.

[9] Bansal, R. and Basar, T., "Stochastic teams with non-classical information revisited: When is an affine law optimal," *IEEE Trans. on Automatic Control*, Vol. 32, pp. 554-9, 1987.

[10] Bellman, R., *Introduction to matrix analysis*, Philadelphia, PA: SIAM, 1997.

[11] Bryson, A. E. and Ho, Y. C., *Applied optimal control*, Bristol, PA: Taylor & Francis, 1975.

[12] Bucy, R. S., "Information and filtering", *Information Sciences*, Vol. 18, pp. 179-187, 1979.

[13] Castanon, D. A. and Sandell, Jr., N. R., “Signaling and uncertainty: a case study”, *Proceedings of CDC 1978*, pp. 1140-1144, 1978.

[14] Chuang, J., C.-I., and Sollenberger, N. R., ”Performance of autonomous dynamic channel assignment and power control for TDMA/FDMA wireless access,” *IEEE J. Select. Areas in Comm.*, Vol. 12, No. 8, pp. 1314-23, October 1994.

[15] Cohen, M. L., “The Fisher information and convexity”, *IEEE Trans. on Information Theory*, Vol. 14, pp. 591-592, 1968.

[16] Cover, T. M. and Thomas, J. A., *Elements of information theory*, New York: Wiley, 1991.

[17] Cox, D. C., and Reudink, D. O., ”Dynamic channel assignment in high capacity mobile communication systems,” *Bell Syst. Tech. J.,* pp. 1833-57, August 1971.

[18] Dahleh, M. A., and Diaz-Bobillo, I. J., *Control of uncertain systems, a linear programming approach*, Englewood Cliffs, NJ: Prentice-Hall, 1995.

[19] Deng, M., and Ho, Y.-C., ”Sampling-selection method for stochastic optimization problems,” *Automatica*, Vol. 35, No. 2, pp. 331-8, 1999.

[20] Dziong, Z., Jia, M., and Mermelstein, P., “Adaptive traffic admission for integrated services in CDMA wireless-access networks,” *IEEE J. Select. Areas in Comm.*, Vol. 14, No. 9, pp. 1737-47, December 1996.

[21] El-Osery, A. and Abdallah, C., ”Distributed power control in CDMA cellular systems,” *IEEE Antennas and Propagation Magazine*, Vol. 42, No. 4, pp. 152-9, August 2000.

[22] Eyre, J. and Yanakiev, D. and Kanellakopoulos, I., “A simplified framework for string stability analysis of automated vehicles,” *Vehicle System Dynamics*, Vol. 30, No. 5, pp. 375-405, Swets & Zeitlinger, November 1998.

[23] Fan, C. and Speyer, J. L. and Jaensch, C. R., “Centralized and decentralized solutions of the linear exponential Gaussian problem,” *IEEE Trans. on Automatic Control*, Vol. 39, pp. 1986-2003, 1994.

[24] Feng, X. and Loparo K. A. and Fang, Y., “Optimal state estimation for stochastic systems: An information theoretic approach,” *IEEE Trans. on Automatic Control*, Vol. 42, pp. 771-785, 1997.

[25] Foschini, G. J. and Miljanic, Z., "A simple distributed autonomous power control algorithm and its convergence," *IEEE Trans. on Vehicular Technology*, Vol. 42, No. 4, pp. 641-646, November 1993.

[26] Foschini, G. J., and Miljanic, Z., "Distributed autonomous wireless channel assignment algorithm with power control," *IEEE Trans. on Vehicular Technology*, Vol. 44, No. 3, pp. 420-9, August 1995.

[27] Goldsmith, A. J., Greenstein, L. J., and Foschini, G. J., "Error statistics of real-time power measurements in cellular channels with multipath and shadowing," *IEEE Trans. on Vehicular Technology*, Vol. 43, No. 3, pp. 439-46, August 1994.

[28] Goodman, D. J., Grandhi, S. A., and Vijayan, R., "Distributed dynamic channel assignment schemes", in *Proc. IEEE Vehicular Technology Conf. (VTC),* , pp. 532-35, 1993.

[29] Grandhi, S. A., Vijayan, R. and Goodman, D. J., "Distributed power control in cellular radio systems," *IEEE Trans. on Comm.*, Vol. 42, No. 2, pp. 226-8, February-April 1994.

[30] Gudmundson, M., "Correlation model for shadow fading in mobile radio systems," *Electronics Letters*, Vol. 27, No. 23, pp. 2145-6, November 1991.

[31] Gunnarsson, F., *Power control in cellular radio systems: analysis, design and estimation*, Ph.D. Dissertation, Dept. of Electrical Engineering, Linkoping University, Sweden, 2000.

[32] Hanly, S. V., "Congestion measures in DS-CDMA networks," *IEEE Trans. on Comm.*, Vol. 47, No. 3, pp. 426-437, March 1999.

[33] Hansen, C. J., *Probing techniques for multiuser channels with power control*, Ph.D. Dissertation, Electrical Engineering Dept., UCLA, 1997.

[34] Hansen, C. J., and Pottie, G. J., "A distributed access algorithm for cellular radio systems with channel partitioning," *IEEE Trans. on Vehicular Technology*, Vol. 48, No. 1, pp. 76-82, January 1999.

[35] Herdtner, J. D., and Chong, E. K. P., "Analysis of a class of distributed asynchronous power control algorithms for cellular wireless systems," *IEEE J. Select. Areas in Comm.*, Vol. 18, No. 3, pp. 436-46, March 2000.

[36] Ho, Y.-C. and Chu, K.C., "Team decision theory and information structures in optimal control problems - Parts I and II," *IEEE Trans. on Automatic Control*, Vol. 17, pp. 15-22,22-28, 1972.

[37] Ho, Y.-C. and Kastner, M.P. and Wong, E., “Teams, signaling and information theory,” *IEEE Trans. on Automatic Control*, Vol. 23, pp. 305-311, 1978.

[38] Ho, Y.-C., “Team decision theory and information structures,” *Proceedings of the IEEE*, Vol. 68, pp. 644-654, 1980.

[39] Ho, Y.-C. and Chu, K.C., “Information structure in dynamic multi-person control problems,” *Automatica*, Vol. 10, pp. 341-351, 1974.

[40] Ho, Y.-C. and Chang, T. S., “Another look at the non-classical information structure problem,” *IEEE Trans. on Automatic Control*, Vol. AC-25, pp. 537-540, 1980.

[41] Horn, R. A. and Johnson, C. R., *Matrix analysis*, Cambridge: Cambridge University Press, 1985.

[42] Howard, R. A., “Information value theory,” *IEEE Trans. on Systems Sciences and Cybernetics*, Vol. SSC-2, pp. 22-34, 1966.

[43] Jakes, W.C., *Microwave mobile communications*, New York, NY: John Wiley & Sons, 1974.

[44] Jaynes, E. T., “Information theory and statistical mechanics,” *Phys. Rev.*, Vol. 106, pp. 620-630, 1957.

[45] Kalata, P. and Priemer, R., “Linear prediction, filtering and smoothing: An information theoretic approach,” *Information Sciences*, Vol. 17, pp. 1-14, 1979.

[46] Kandukuri, S., and Boyd, S., ”Optimal power control in interference limited fading wireless channels with outage probability specifications,” Submitted to *IEEE J. Select. Areas in Comm.*, May 2000.

[47] Katzela, I., and Naghshineh, M., ”Channel assignment schemes for cellular mobile telecommunication systems: A comprehensive survey,” *IEEE Personal Communications Magazine*, Vol. 3, pp. 10-31, June 1996.

[48] Khalil, H. K., *Nonlinear systems*, Second Edition, Upper Saddle River, NJ: Prentice-Hall, 1996.

[49] Lee, J. T. and Lau, E. and Ho, Y.-C., ”On the global optimum of the Witsenhausen counterexample,” *Draft version*, 1998.

[50] Lee, J. T. and Lau, E. and Ho, Y.-C., ”The Witsenhausen counterexample: A hierarchical search approach for nonconvex optimization problems,” *IEEE Trans. on Automatic Control*, Vol. 46, No. 3, pp. 382-397, March 2001.

[51] Leung, K. K., "A Kalman Filter method for power control in broadband wireless networks," in *Proc. of INFOCOM'99* , pp. 948-56, 1999.

[52] Leung, Y.-W., "Power control in cellular networks subject to measurement error," *IEEE Trans. on Comm.*, Vol. 44, No. 7, pp. 772-5, July 1996.

[53] Lozano, A., and Cox, D. C., "Integrated dynamic channel assignment and power control in TDMA mobile wireless communication systems," *IEEE J. Select. Areas in Comm.*, Vol. 17, No. 11, pp. 2031-40, November 1999.

[54] Ludwin, W., and Chlebus, E., "Performance comparison of simulators for cellular mobile networks," *Applied Mathematical Modeling*, Vol. 20, No. 8, pp. 585-7, Elsevier, August 1996.

[55] Marschak, J., "Elements for a theory of teams," *Management Science*, Vol. 1, pp. 127-137, 1955.

[56] Meier, L. and Larson, R. E. and Tether, A. J., "Dynamic programming for stochastic control of discrete systems," *IEEE Trans. on Automatic Control*, Vol. 16, pp. 767-775, 1971.

[57] Narasimhan, R., and Cox, D. C., "Speed estimation in wireless systems using wavelets," *IEEE Trans. on Comm.*, Vol. 47, No. 9, pp. 1357-64, September 1999.

[58] Ooi, J. M. and Verbout, S. M. and Ludwig, J. T. and Wornell, G. W., "A separation theorem for periodic sharing information patterns in decentralized control," *IEEE Trans. on Automatic Control*, Vol. 42, pp. 1546-1550, 1997.

[59] Papadimitriou, C. H. and Tsitsiklis, J. N., "Intractable problems in control theory," *SIAM Journal on Control and Optimization*, Vol. 24, pp. 639-654, 1986.

[60] Pora, W., Chambers, J. A., and Constantinides, A. G., "Combination of Kalman filter and constant modulus algorithm with variable step-size for equalization of fast fading channels," *Electronics Letters*, Vol. 34, No. 18, pp. 1718-19, September 1998.

[61] Radner, R., "Team decision problems," *Ann. Math. Stat.*, Vol. 33, pp. 857-881, 1962.

[62] Radner, R. and Marschak, J., *Economic theory of teams*, New Haven, CT: Yale University Press, 1972.

[63] Sandell, Jr., N.R. and Athans, M., "Solution of some non-classical LQG stochastic decision problems," *IEEE Trans. on Automatic Control*, Vol. 19, pp. 108-116, 1974.

[64] Saridis, G. N., "Entropy formulation of optimal and adaptive control," *IEEE Trans. on Automatic Control*, Vol. 33, pp. 713-721, 1988.

[65] Shannon, C. E., "A mathematical theory of communication," *Bell System Technical Journal*, Vol. 27, pp. 379-423, 623-656, 1948.

[66] Sheridan, T. B., "Reflections on information and information value," *IEEE Trans. on Systems, Man and Cybernetics*, Vol. 25, pp.194-196, 1995.

[67] Shoarinejad, K., Speyer, J. L., and Pottie, G. J., "Feasibility and global stability of power control algorithms for cellular wireless systems," Submitted to *IEEE Trans. on Comm.*, 2001.

[68] Song, L., Mandayam, N. B., and Gajic, Z., "Analysis of an up/down power control algorithm for the CDMA reverse link under fading," Submitted to *IEEE J. Select. Areas in Comm.*, 2000.

[69] Sorenson, H. W., Ed., *Kalman filtering: theory and application*, New York, NY: IEEE Press, 1985.

[70] Speyer, J. L., "Computation and transmission requirements for a decentralized Linear Quadratic Gaussian control problem," *IEEE Trans. on Automatic Control*, Vol. 24, pp. 266-269, 1979.

[71] Speyer, J. L., "A system theoretic approach to autonomous vehicle dynamic formation," *A research proposal to Air Force Office of Scientific Research*, 1996.

[72] Stuber, G., L., *Principles of mobile communications*, Norwell, MA: Kluwer Academic Publishers, 1996.

[73] Teneketzis, D., and Andersland, M. S., "On partial order characterization of information structures," *Mathematics of Control, Signals and Systems*, Vol. 13, No. 4, pp. 277-92, Springer-Verlag, 2000.

[74] Tsai, Y. A. and Casiello, F. A. and Loparo, K. A., "Discrete-time entropy formulation of optimal and adaptive control problems," *IEEE Trans. on Automatic Control*, Vol. 37, pp. 1083-1088, 1992.

[75] Tsoukatos, K. P., "Power control in a mobility environment," in *Proc. IEEE Vehicular Technology Conf. (VTC),* pp. 740-744, 1997.

[76] Ulukus, S., and Yates, R. D., "Stochastic power control for cellular radio systems," *IEEE Trans. on Comm.*, Vol. 46, No. 6, pp. 784-98, June 1998.

[77] Varaiya, P. and Walrand, J., "On delayed sharing patterns," *IEEE Trans. on Automatic Control*, Vol. AC-23, pp. 443-445, 1978.

[78] Verdone, R., and Zanella, A., "On the optimization of fully distributed power control techniques in cellular radio systems," *IEEE Trans. on Vehicular Technology*, Vol. 49, No. 4, pp. 1440-48, July 2000.

[79] Viterbi, A., J., Viterbi, A. M., and Zehavi, E., "Performance of power-controlled wide-band terrestrial digital communication," *IEEE Trans. on Comm.*, Vol. 41, No. 4, pp. 559-569, April 1993.

[80] Wang, C. C-Y., *Power control strategies and variable bit allocation for FH-CDMA wireless systems*, Ph.D. Dissertation, Electrical Engineering Dept., UCLA, 1996.

[81] Wang, S. H., and Davison, E. J., "On the stabilization of decentralized control systems," *IEEE Trans. on Automatic Control*, Vol. 18, pp. 473-8, 1973.

[82] Weidemann, H.L. and Stear, E. B., "Entropy analysis of estimating systems," *IEEE Trans. on Information Theory*, Vol. 16, pp. 264-270, 1970.

[83] Weidemann, H.L., "Entropy analysis of feedback control systems," in *Advances in Control Systems*, C. Leondes Ed., New York: Academic, 1969.

[84] Witsenhausen, H.S., "A counterexample in stochastic optimal control," *SIAM Journal of Control*, Vol. 6, pp. 131-147, 1968.

[85] Witsenhausen, H.S., "Separation of estimation and control for discrete time systems," *Proceedings of the IEEE*, Vol. 59, pp. 1557-1566, 1971.

[86] Yanakiev, D. and Kanellakopoulos, I., "Speed tracking and vehicle follower control design for heavy-duty vehicles," *Vehicle System Dynamics*, Vol. 25, pp. 251-276, Swets & Zeitlinger, 1996.

[87] Yates, R. D., "A framework for uplink power control in cellular radio systems," *IEEE J. Select. Areas in Comm.*, Vol. 13, No. 7, pp. 1341-7, September 1995.

[88] Yates, R. D., and Ching-Yao Huang, "Integrated power control and base station assignment," *IEEE Trans. on Vehicular Technology*, Vol. 44, No. 3, pp. 638-44, August 1995.

[89] Zander, J., "Performance of optimum transmitter power control in cellular radio systems," *IEEE Trans. on Vehicular Technology*, Vol. 41, No. 1, pp. 57-62, February 1992.

[90] Zander, J., "Distributed co-channel interference control in cellular radio systems," *IEEE Trans. on Vehicular Technology*, Vol. 41, No. 3, pp. 305-311, August 1992.

[91] Zhou, K., Doyle, J. C., and Glover, K., *Robust and optimal control*, Upper Saddle River, NJ: Prentice-Hall, 1995.